UNIVERSITY OF WAIKATO

Hamilton
New Zealand

Modelling the Yield Curve with

Orthonomalised Laguerre Polynomials:

An Intertemporally Consistent Approach

with an Economic Interpretation

Leo Krippner

Department of Economics

Working Paper in Economics 1/03

September 2003

Leo Krippner
Reserve Bank of New Zealand
P.O. Box 2498
Wellington, New Zealand

Tel: +64 (04) 471-3605
Fax: +64 (07) 471-3800
Email: krippnerl@rbnz.govt.nz
http://www.rbnz.govt.nz

Abstract

This article provides theoretical foundations for the popular orthonormalised Laguerre polynomial (OLP) model of the yield curve, as originally introduced by Nelson and Siegel (1987). Intertemporal consistency is provided by deriving the volatility-adjusted OLP (VAO) model of the yield curve using the risk-neutral Heath, Jarrow and Morton (1992) framework, and including an allowance for term premia as noted in Duffee (2002). An economic interpretation is provided by deriving the relationship between the VAO model and the Berardi and Esposito (1999) yield curve model that is based on a generic general equilibrium model of the economy. In empirical applications using almost 50 years of United States data, the VAO model outperforms the random walk when used to forecast the yield curve out of sample, and the level of the yield curve as measured by the VAO model is shown to be cointegrated with CPI inflation, as predicted.

Keywords

JEL Classification

Acknowledgements

1   Introduction

Nelson and Siegel (1987) proposed the original othonormalised Laguerre polynomial (OLP) model of the yield curve. This approach, which is sometimes referred to and expressed in the linearly equivalent exponential-polynomial form, has subsequently been extended and revisited in Svensson (1994), Hunt (1995), Bliss (1997), Mansi and Phillips (2001), Diebold and Li (2002), and Krippner (2002). OLP models are widely used by researchers and market practitioners and perform very favourably in comparison with other yield curve models.[1]

Notwithstanding their popularity, OLP models have two theoretical shortcomings. The first is that OLP models fitted sequentially to cross-sectional yield curve data cannot be intertemporally consistent, as identified in Björk and Christensen (1999), Filopović (1999a), and Filopović (1999b). This leaves some researchers wary about using the estimated cross-sectional coefficients of OLP models in a time-series context. The second theoretical shortcoming is that OLP models lack a fundamental economic foundation. That is, there has been no formal attempt in the literature to relate the parameters and coefficients of OLP models back to economic state variables, as is the basis for equilibrium models of the yield curve.[2] This absence of "economic meaning" leaves some

---

[1] Bank for International Settlements (1999) notes that ten central banks (of twelve surveyed) routinely use either the Nelson and Siegel (1987) and/or the Svensson (1994) model as their primary method for analysing the yield curve. Other examples of the practical application of exponential-polynomial models are Kacala (1993), Barrett, Gosnell and Heuson (1995), Schich (1997), Söderlind and Svensson (1997), Brooks and Yong Yan (1999), Monetary Authority of Singapore (1999), Soto (2001), Schmid and Kalemanova (2002), and Fang and Muljono (2003). For a comparison to other models, see Dahlquist and Svensson (1996), Seppala and Viertio (1996), Bliss (1997), Fergusson and Raymar (1998), Ioannides (2003), and Jordan and Mansi (2003).

[2] Vasicek (1977) and Cox, Ingersoll and Ross (1985b) are examples of equilibrium models of the yield curve.

researchers wary about interpreting the parameters derived from OLP models in conjunction with economic variables. Together, these factors have resulted in OLP models being restricted mainly to cross-sectional applications, i.e "yield curve fitting".

The theoretical work in this article directly addresses the points raised in the previous paragraph. Specifically, section 2 derives the volatility-adjusted OLP (VAO) model of the forward rate curve using the Heath, Jarrow and Morton (1992) framework, and including an allowance for term premium effects as noted by Duffee (2002). The VAO model is cross-sectionally and intertemporally consistent by construction, and this dual consistency is exploited to derive a model for forecasting the yield curve using current yield curve data. Section 3 derives the relationship between the VAO model and the Berardi and Esposito (1999) model of the forward rate curve that is based on a generic general equilibrium model of the economy. This allows an interpretation of the VAO model in terms of economic state variables.

The empirical application of the VAO model is to United States yield curve and inflation data over the period 1954 to 2003. Section 5 investigates the goodness-of-fit and the predicted time series properties of the VAO model applied to the full sample of yield curve data, section 6 investigates the ability of the VAO model to forecast the yield curve out-of-sample, and section 7 investigates the predicted relationship between inflation and the level of the yield curve as measured by the VAO model.

2 The volatility-adjusted orthonormalised Laguerre polynomial model of the forward rate curve

Section 2 proceeds as follows: section 2.1 defines the framework and terminology used to derive the VAO model of the forward rate curve, section 2.2 derives the VAO model, section 2.3 makes several observations about the VAO model, and section 2.4 explicitly derives the intertemporal relationship implied by the VAO model.

2.1 A risk-neutral framework and the term premium function

The foundation for a cross-sectionally and intertemporally consistent model of the entire forward rate curve is the generic risk-neutral relationship provided by the Heath, Jarrow and Morton (1992) (hereafter HJM) framework. From equation 26 of the HJM paper, the risk-neutral relationship between the forward rate curve and the process for the short rate is:

$$f(0,m) = r(m) - \sum_{n=1}^{N} \int_{0}^{m} \alpha_n(s,m)\,ds - \sum_{n=1}^{N} \int_{0}^{m} \sigma_n(s,m)\,d\tilde{W}_n(s) \quad (1)$$

where:

- $f(0,m)$ is the forward rate curve at the initial time, i.e instantaneous forward rates as a function of maturity m $(m \geq 0)$;

- $r(m) = f(m,m)$ is the path of the instantaneous short rate as a function of "future time"/maturity m;

- N is is the number of independent stochastic processes that effect instantaneous random shocks to the forward rate curve and the short rate;

- $\alpha_n(s, m) = \sigma_n(s, m) \left[\int_s^m \sigma_n(s, u) \, du \right]$ is the drift component for the forward rate curve/short rate process n (u is a dummy integration variable for m, and s is a dummy integration variable for time t as discussed in section 2.4);

- $\sigma_n(s, m)$ is the volatility function for the forward rate curve/short rate process n; and

- $d\tilde{W}_n(s)$ are independent Wiener variables under the risk-neutral measure.

Applying the expectations operator as at the initial time, i.e E_0, to equation 1 provides an explicit relationship between $f(0, m)$ and $E_0[r(m)]$, the expected path of the short rate at the initial time, i.e:[3]

$$f(0, m) = E_0[r(m)] - \sum_{n=1}^{N} \int_0^m \alpha_n(s, m) \, ds \qquad (2)$$

Of course, the forward rate curve observed in the "real world" (i.e under the physical measure) will not necessarily conform to the risk-neutral processes of equations 1 and 2; the market price or yield of each physical interest rate instrument may also include premia to compensate for the various risks associated with holding that instrument. The main compensation is for expected interest rate volatility itself, but additional factors may require compensation, such as inflation/monetary policy regime risks, liquidity risks etc. The approach

---

[3] The expectation of the stochastic term in equation 1 is zero; see Ross (1997) pp. 541-542.

in this article is to collect the physical expression of all risks into a single "term premium function" $\delta(m)$ and add this to equation 2, i.e:

$$f(0, m) = \delta(m) + E_0[r(m)] - \sum_{n=1}^{N} \int_0^m \alpha_n(s, m) \, ds \qquad (3)$$

This approach is conceptually similar to the "essentially affine" models of Duffee (2002), in that the shape of the forward rate curve is allowed to contain components that are independent of the expected evolution of the short rate. Equation 3 shows that the forward rate curve is defined by the term premium, the expected path of the short rate, and the volatility structure for the forward rate curve/short rate that defines the "volatility adjustment" term $\sum_{n=1}^{N} \int_0^m \alpha_n(s, m) \, ds$. The following section specifies structures for these three components, and thereby defines the forward rate curve.

2.2 The volatility-adjusted OLP model of the forward rate curve

The literature on OLP models typically starts from a forward rate curve specified as $f(m) = \beta_1 + \beta_2 \cdot [-\exp(-\phi m)] + \beta_3 \cdot [-\exp(-\phi m)(-2\phi m + 1)]$,[4] where ϕ is a fixed positive constant that alters the rate of the exponential decay, and β_n are linear coefficients. Krippner (2002) generalises this functional form by proposing a linear combination of OLP "modes",[5] which allows the

---

[4]For example, the models of Nelson and Siegel (1987), Hunt (1995), Diebold and Li (2002) are linearly equivalent to this specification. The models of Svensson (1994), Bliss (1997), and Mansi and Phillips (2001) include additional terms with a different exponential decay rate, but are analogous to the OLP specification in this article.

[5]Courant and Hilbert (1953) pp. 93-97, or Rainville and Bedient (1981) pp. 395-396, contain more information on OLP functions. They are a series of solutions to the second-order differential equation noted in Courant and Hilbert (1953) pp. 328-331, and members of such solution sets are commonly referred to as "modes".

OLP model be extended as desired. The first three OLP modes are illustrated in figure 1, and are intuitively named the Level, Slope, and Bow modes based on their shapes. However, the results in Björk and Christensen (1999), Filopović (1999a), Filopović (1999b), and the discussion of section 2.3 of this article show that OLP specifications of the forward rate curve cannot be intertemporally consistent.

[Figure 1 here]

Rather, an intertemporally consistent model of the forward rate curve can be constructed using an OLP specification for $\delta\left(m\right)+E_0\left[r\left(m\right)\right]$ within equation 3. This article uses the first three OLP modes to develop the model used for the empirical work, i.e:

$$\delta\left(m\right)+E_0\left[r\left(m\right)\right]=\beta_1+\beta_2\cdot\left[-\exp\left(-\phi m\right)\right]+\beta_3\cdot\left[-\exp\left(-\phi m\right)\left(-2\phi m+1\right)\right] \quad (4)$$

Note that each β_n is a composite coefficient, i.e $\beta_n=\gamma_n+\lambda_n$, where γ_n is the term premium component, and λ_n is associated with the expected evolution of the short rate. For convenience γ_n is assumed to be constant, and λ_n is assumed to change with a deterministic and stochastic component as time evolves.[6] The evolution of the deterministic components of λ_n is derived in section 2.4, and the most analytically tractable representation of the stochastic component of

---

[6] A time-varying term premium could readily be allowed for, but would later require extra terms to allow for the effects of term premium volatility. Empirically, the assumption of a constant term premium implicitly (and reasonably) assumes that any time-varying components of the term premium are slow-moving and/or regime dependent, and the dynamics of the yield curve are primarily due to changes in the expected short rate.

λ_n (and hence β_n) is an independent and constant Gaussian distribution, i.e $d\beta_n = \sigma_n \cdot d\tilde{W}_n(s)$ with $\sigma_n > 0$ and $d\tilde{W}_n(s) \sim N(0,1)$, and $\text{cov}(d\beta_n, d\beta_p) = 0$ for $n \neq p$.[7] This means the volatility adjustment term $\sum_{n=1}^{3} \int_0^m \alpha_n(s,m) ds$ can be written as $\sum_{n=1}^{3} \sigma_n^2 \cdot h_n(\phi, m)$, and it remains to calculate the functional form for each $h_n(\phi, m)$.

The results for the first two modes are available in the literature as $h_1(\phi, m) = \frac{1}{2}m^2$, and $h_2(\phi, m) = \frac{1}{2\phi^2}[1 - \exp(-\phi m)]^2$.[8] The generic expression for $h_n(\phi, m)$ when $n > 1$ is derived in Appendix A, and the result for the third mode is: $h_3(\phi, m) = \frac{1}{2\phi^2}[1 - \exp(-\phi m)]^2 - \frac{1}{\phi^2}[1 - \exp(-\phi m) - \phi m \exp(-\phi m)]^2$. These functions are illustrated in figure 2, and may be interpreted as (time-invariant) volatility adjustments as a function of maturity per unit of variance in $d\beta_n$.

[Figure 2 here]

Substituting equation 4 and the volatility adjustment results for each of the three modes into equation 3 gives the three-mode volatility-adjusted OLP (VAO) model of the forward rate curve that is used in section 5 to fit cross-sections of yield curve data, i.e:

$$
\begin{aligned}
f(0,m) &= \beta_1 + \beta_2 \cdot [-\exp(-\phi m)] + \beta_3 \cdot [-\exp(-\phi m)(-2\phi m + 1)] \\
&\quad - \sigma_1^2 \cdot h_1(\phi, m) - \sigma_2^2 \cdot h_2(\phi, m) - \sigma_3^2 \cdot h_3(\phi, m)
\end{aligned}
\tag{5}
$$

---

[7]These assumptions are investigated and discussed in section 5.2. Note that instantaneous stochastic changes to $E_0[r(m)]$ will be of the form: $dE_0[r(m)] = \sigma_1 \cdot d\tilde{W}_1(s) + \sigma_2 \cdot [-\exp(-\phi m)] \cdot d\tilde{W}_2(s) + \sigma_3 \cdot [-\exp(-\phi m)(-2\phi m + 1)] \cdot d\tilde{W}_3(s)$.

[8]See HJM pp. 90-92, or de La Grandville (2001) pp. 368-372. Note that HJM uses the Slope volatility function $\exp(-\phi m/2)$, so there is a scalar difference between the HJM result and $h_2(\phi, m)$ as presented in this article.

2.3 Observations about the VAO model

Firstly, the results of section 2.2 confirm the result that the forward rate curves specified with OLP functions cannot be intertemporally consistent within a risk-neutral setting; the HJM framework specifies that volatility in those OLP functions will lead to functions with form $\exp\left(-2\phi m\right) \cdot (4\phi m)^n$ that are being ignored in the OLP specification of the forward rate curve.[9] This is the essence of the results in Björk and Christensen (1999), Filopović (1999a), and Filopović (1999b).

Secondly, the addition of each $h_n(\phi, m)$ function within the VAO model may be seen as a "manifold expansion" analogous to that suggested by Björk and Christensen (1999) pp. 338-339 to make the Nelson and Siegel (1987) model consistent with the risk-neutral Hull and White (1990) model. The m^2 term in the VAO model is a "manifold expansion" to account for the effect of volatility in the Level mode, and this term also occurs in other risk-neutral models that incorporate constant forward rate volatility for all maturities.[10]

Thirdly, the VAO model is of the no-arbitrage class in the sense noted by Brandt and Yaron (2002). That is, if a precise fit to market-observed data is required, then the number of modes may in principle be increased to equal the number of instruments. However, as noted by Brandt and Yaron (2002), the user will often prefer a more parsimonious representation (i.e an approx-

---

[9] Except in the (unrealistic) completely deterministic case. That is, with zero volatility, the volatility adjustment would equal zero, and the initial forward rate curve and the expected path of the short rate would be identical.

[10] For example, the Vasicek (1977) model with zero mean-reversion (as noted in Hull 2000 p. 567), the Ho and Lee (1986) model (as noted in Hull 2000, pp. 108 and 572-574), and the HJM constant volatility model (pp. 90-91).

imately arbitrage-free model) in conjunction with some well-behaved yield or price residuals for tractibility, intertemporal consistency, and to allow for the fact that market-observed data will inevitably contain "measurement errors".[11]

2.4 The time evolution of the VAO model coefficients

The HJM framework presented in section 2.1 implicitly assumes that the forward rate curve is observed at time $t = 0$, and expectations of the path of the short rate are as at time $t = 0$. Of course, equation 1 (and its subseqeuent derivations) also applies to the forward rate curve at time t, i.e $f(t, m)$, and the expected path of the short rate as at time t, i.e $E_t[r(t + m)]$, where E_t represents the expectations operator applied at time t (i.e expectations are formed using information available up to time t). Introducing a time-increment τ (> 0) to denote a finite evolution in time from t to $t + \tau$ allows the explicit derivation of the intertemporal relationship between the expected path of the short rate at times t and $t + \tau$ within the HJM framework. This derivation is contained in Appendix B, with the essential result that:

$$E_{t+\tau}[r(t + \tau + m)] = E_t[r(t + \tau + m)] + \sum_{n=1}^{N} \int_{t}^{t+\tau} \sigma_n(s, m) \, d\tilde{W}_n(s) \quad (6)$$

For the following, it is convenient to express the expected path of the short rate from section 2.2 in vector form, i.e: $E_t[r(t + m)] = [\boldsymbol{\lambda}(t)]' \mathbf{g}(\phi, m)$,

---

[11]The generic OLP specification in Krippner (2002), and the corresponding results for $h_n(\phi, m)$ as derived in Appendix A, allows the user to extend the VAO model to give the precision versus parsimony trade-off that best suits their particular application. Such extensions remain to be investigated in future work.

where $\mathbf{g}(\phi, m) = \{1, -\exp(-\phi m), -\exp(-\phi m)(-2\phi m + 1)\}'$ and $\boldsymbol{\lambda}(t) = \{\lambda_1(t), \lambda_2(t), \lambda_3(t)\}'$. The stochastic term $\sum_{n=1}^{N} \int_t^{t+\tau} \sigma_n(s, m) d\tilde{W}_n(s)$ will also be of the form $[\boldsymbol{\varepsilon}(\phi, \tau)]' \mathbf{g}(\phi, m)$, as noted in Appendix B, and representing all of the terms in equation 6 in this vector form results in the equality:

$$[\boldsymbol{\lambda}(t + \tau)]' \mathbf{g}(\phi, m) = [\boldsymbol{\lambda}(t)]' \mathbf{g}(\phi, \tau + m) + [\boldsymbol{\varepsilon}(\phi, \tau)]' \mathbf{g}(\phi, m) \qquad (7)$$

Introducing the coefficient matrix:

$$\bullet \ [\boldsymbol{\Phi}(\phi, \tau)]' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \exp(-\phi\tau) & 0 \\ 0 & -2\phi\tau \exp(-\phi\tau) & \exp(-\phi\tau) \end{bmatrix} ;$$

it may be verified directly that $\mathbf{g}(\phi, \tau + m) = [\boldsymbol{\Phi}(\phi, \tau)]' \mathbf{g}(\phi, m)$. This enables the evolution of the expected path of the short rate over the time step τ to be written as $[\boldsymbol{\lambda}(t + \tau)]' \mathbf{g}(\phi, m) = [\boldsymbol{\lambda}(t)]' [\boldsymbol{\Phi}(\phi, \tau)]' \mathbf{g}(\phi, m) + [\boldsymbol{\varepsilon}(\phi, \tau)]' \mathbf{g}(\phi, m)$. Factoring out the common term $\mathbf{g}(\phi, m)$, and then taking the transpose gives the following result (in column-vector form):

$$\boldsymbol{\lambda}(t + \tau) = \boldsymbol{\Phi}(\phi, \tau) \boldsymbol{\lambda}(t) + \boldsymbol{\varepsilon}(\phi, \tau) \qquad (8)$$

As noted in section 2.2, the actual estimates of $\boldsymbol{\beta}(t) = \{\beta_1(t), \beta_2(t), \beta_3(t)\}'$ will also contain a term premium component $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, \gamma_3\}'$, i.e $\boldsymbol{\beta}(t) = \boldsymbol{\gamma} + \boldsymbol{\lambda}(t)$. Hence, the equation for the time evolution of the estimated coefficients $\boldsymbol{\beta}(t)$ will also require a constant vector, denoted here as $-\boldsymbol{\mu}$, to allow for the term premium:[12]

---

[12] Written in full, equation 9 is $\boldsymbol{\gamma} + \boldsymbol{\lambda}(t + \tau) = \boldsymbol{\gamma} + \boldsymbol{\Phi}(\phi, \tau) \boldsymbol{\lambda}(t) + \boldsymbol{\varepsilon}(\phi, \tau) =$

$$\boldsymbol{\beta}\left(t+\tau\right)=-\boldsymbol{\mu}+\boldsymbol{\Phi}\left(\phi,\tau\right)\boldsymbol{\beta}\left(t\right)+\boldsymbol{\varepsilon}\left(\phi,\tau\right) \tag{9}$$

Equation 9 is a first-order vector autoregression (VAR), and may be used to predict the time-series properties of the estimated coefficients in section 5. Firstly, the eigenvalues of $\boldsymbol{\Phi}\left(\phi,\tau\right)$ are $\{1,\exp\left(-\phi\tau\right),\exp\left(-\phi\tau\right)\}$, suggesting there is a single unit root in the VAR. The coefficient of 1 in the top-left entry of $\boldsymbol{\Phi}\left(\phi,\tau\right)$ and the block diagonal independence of that entry further identifies that the unit root should be associated with $\beta_1\left(t\right)$. In the remaining sub-matrix, the repeated eigenvalues $\exp\left(-\phi\tau\right)$ are less than 1, and so $\beta_2\left(t\right)$ and $\beta_3\left(t\right)$ should both be mean-reverting and therefore stationary.

The relationship between the expected path of the short rate and the forward rate curve from equation 5 provides the link to forecasting the forward rate curve (and hence the yield curve) from the current yield curve, i.e:

$$
\begin{aligned}
E_t\left[f(t+\tau,m)\right] &= E_t\left\{\left[-\boldsymbol{\mu}+\boldsymbol{\beta}\left(t+\tau\right)\right]'\right\}\mathbf{g}(\phi,m)-\mathbf{v}'\mathbf{h}(\phi,m) && \text{(10a)} \\
&= \left[-\boldsymbol{\mu}+\boldsymbol{\Phi}\left(\phi,\tau\right)\boldsymbol{\beta}\left(t\right)\right]'\mathbf{g}(\phi,m)-\mathbf{v}'\mathbf{h}(\phi,m) && \text{(10b)}
\end{aligned}
$$

where $\mathbf{v}=\left\{\sigma_1^2,\sigma_2^2,\sigma_3^2\right\}'$ and $\mathbf{h}(\phi,m)=\{h_1\left(\phi,m\right),h_2\left(\phi,m\right),h_3\left(\phi,m\right)\}'$ for the three-mode VAO model, and $\mathbf{v}'\mathbf{h}(\phi,m)$ is time-invariant. The application of equation 10b to forecasting the yield curve is contained in section 6.

$\left[\mathbf{I}-\boldsymbol{\Phi}\left(\phi,\tau\right)\right]\boldsymbol{\gamma}+\boldsymbol{\Phi}\left(\phi,\tau\right)\left[\boldsymbol{\gamma}+\boldsymbol{\lambda}\left(t\right)\right]+\boldsymbol{\varepsilon}\left(\phi,\tau\right)$, and so $-\boldsymbol{\mu}=\left[\mathbf{I}-\boldsymbol{\Phi}\left(\phi,\tau\right)\right]\boldsymbol{\gamma}$.

3 An economic interpretation of the VAO model

Section 3 proceeds as follows: section 3.1 summarises a generic general equilibrium approach to modelling the yield curve, and sections 3.2 and 3.3 respectively show that the real and inflation components of the expected path of the short rate from that model are naturally approximated by OLP modes. From these results, section 3.4 discusses the economic interpretation of the generic VAO model parameters and coefficients.

3.1 A generic general equilibrium approach to modelling the yield curve

Berardi and Esposito (1999) (hereafter BE) derives a generic affine multifactor model of the forward rate curve from a general equilibrium model based on the economic model proposed by Cox, Ingersoll and Ross (1985a). The BE approach encapsulates all Vasicek-type and Cox-Ingersoll-Ross-type equilibrium models,[13] and many other equilibrium models that have been proposed in the literature. It also encapsulates the affine multifactor models of Duffie and Kan (1996) and Dai and Singleton (2000), providing a general equilibrium basis for those models and explicitly accounting for the separation between real and nominal variables. The BE generic risk-neutral J-factor process is:

$$ds_j(t) = \kappa_j \left[\theta_j - s_j(t)\right] dt + \sqrt{\sigma_{0j}^2 + \sigma_{1j}^2 \cdot s_j(t)} \cdot dz_j(t) \tag{11}$$

---

[13] That is, Gaussian and square root dynamics, respectively. See the original article, Vasicek (1977), or Hull (2000) p. 567 for a summary of the Vasicek equilibrium model, and the original article, Cox et al. (1985b), or Hull (2000) p. 570 for a summary of the Cox-Ingersoll-Ross equilibrium model.

where, for $j = 2$ to J:

- $s_j(t)$ are the real state variables, representing returns on factors of production in the economy. These are constructed from the original state variables to be mutually uncorrelated, and will change with a deterministic and stochastic component as time evolves.

- $\kappa_j \, (> 0)$ is the constant mean-reversion coefficient of the process for $s_j(t)$;

- $\theta_j \, (> 0)$ is the constant long-term value of $s_j(t)$;

- $\sqrt{\sigma_{0j}^2 + \sigma_{1j}^2 \cdot s_j(t)}$ is the standard deviation of the stochastic process for $s_j(t)$. The process will be Vasicek-type if $\sigma_{1j} = 0$, Cox-Ingersoll-Ross-type if $\sigma_{0j} = 0$, and can be a mixture of both if σ_{0j} and σ_{1j} are non-zero (with appropriate restrictions to keep $\sigma_{0j}^2 + \sigma_{1j}^2 \cdot s_j(t)$ positive); and

- $dz_j(t)$ are independent Wiener variables.

The $j = 1$ factor is reserved for an inflation state variable, which will be discussed in section 3.3. As noted in Berardi and Esposito (1999), the nominal short rate at any given time is the summation of state variables $s_j(t)$, and for the analysis that follows it is convenient to partition this into inflation and real components, i.e $r(t, s_1, \mathbf{s}) = s_1(t) + \sum_{j=2}^{J} s_j(t)$, where $r(t, s_1, \mathbf{s})$ is the nominal short rate as a function of the inflation state variable $s_1(t)$ and the $(J - 1)$-vector of real state variables $\mathbf{s}(t)$, and $\mathbf{s}(t)$ contributes the real interest rate component $\sum_{j=2}^{J} s_j(t)$.

3.2 The real components of the BE model

The expected path of the real short rate (as distinct from the expected return from a rolling investment in the short rate that is typically used to derive the forward rate curve) may be calculated directly from the expectation of equation 11. That is, applying the expectations operator at time t and using m to denote "future time" from time t gives the relationship: $E_t\left[ds_j\left(t+m\right)\right] = \kappa_j\left[\theta_j - s_j\left(t+m\right)\right]dm$. This is an ordinary differential equation with solution $s_j\left(t+m\right) = \theta_j + A_j \cdot \exp\left(-\kappa_j m\right)$, and the boundary condition at $m = 0$ is $s_j\left(t\right) = \theta_j + A_j$, so $A_j = -\left[\theta_j - s_j\left(t\right)\right]$. Therefore, the real component of the expected path of the short rate may be written as:[14]

$$\sum_{j=2}^{J} s_j\left(t+m\right) = \sum_{j=2}^{J} \theta_j - \sum_{j=2}^{J}\left[\theta_j - s_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right) \qquad (12)$$

To show the correspondence between equation 12 and the OLP functional form, first define ϕ as a central measure of the values of κ_j for $j = 2$ to J, i.e $\phi = \text{central}(\kappa_j)$ (a constant, because κ_j are constants). Hence, $\kappa_j = \phi\left(1 + \Delta_j\right)$ with $-1 < \Delta_j < 1$,[15] and equation 12 may be written equivalently as:

$$\sum_{j=2}^{J} s_j\left(t+m\right) = \sum_{j=2}^{J} \theta_j - \exp\left(-\phi m\right)\cdot\sum_{j=2}^{J}\left[\theta_j - s_j\left(t\right)\right]\cdot\exp\left(-\Delta_j\phi m\right) \qquad (13)$$

Now write each exponential term containing Δ_j as a Taylor expansion

---

[14]This result, and the analogous result for the inflation component in section 3.3, can also be derived using the forward rate curve specified by Berardi and Esposito (1999), and adding the appropriate volatility adjustment calculated using the HJM framework.

[15]This restriction on Δ_j is always possible by construction; in the extreme case, ϕ could be defined as $\max\left(\kappa_j\right)$, and then $-1 < \Delta_j \leq 0 < 1$ (since the lower bound for each κ_j is zero).

around $\Delta_j = 0$ to order $N - 2$, i.e:[16]

$$
\begin{aligned}
\sum_{j=2}^{J} s_j (t + m) &\simeq \sum_{j=2}^{J} \theta_j - \exp(-\phi m) \\
&\quad \times \sum_{j=2}^{J} [\theta_j - s_j(t)] \left[\sum_{n=2}^{N} \frac{1}{(n-2)!} (-\Delta_j \phi m)^{(n-2)} \right] \quad (14a) \\
&= \sum_{j=2}^{J} \theta_j - \exp(-\phi m) \cdot \sum_{n=2}^{N} \omega_n(t) \cdot (\phi m)^{(n-2)} \quad (14b) \\
&= \sum_{j=2}^{J} \theta_j + \sum_{n=2}^{N} \beta_n(t) \\
&\quad \times - \exp(-\phi m) \sum_{k=0}^{n-2} \frac{(-1)^k (n-2)! (2\phi m)^k}{(k!)^2 (n-2-k)!} \quad (14c)
\end{aligned}
$$

where $\omega_n(t)$ in equation 14b is the collection of the coefficients on powers of

$(\phi m)^{(n-2)}$ from the full expansion of the double summation in equation 14a, and

equation 14c is a rearrangement of the summation of exponential-polynomials

into a linearly equivalent summation of OLP functions. This is the generic

OLP form noted in Krippner (2002) and Appendix A of this article, but it

may be verified directly that the $N = 3$ expression of equation 14c is $\beta_1 + \beta_2 \cdot$

$[-\exp(-\phi m)] + \beta_3 \cdot [-\exp(-\phi m)(-2\phi m + 1)]$, as specified in equation 4.

3.3 The inflation component of the BE model

Berardi and Esposito (1999) uses a single independent factor to represent the

inflation rate in the general equilibrium model. For this factor, each of the

---

[16]The residual term $\sum_{n=N+1}^{\infty} \frac{1}{(n-2)!} (-\Delta_j \phi m)^{(n-2)}$ associated with the Taylor expansion approximation will always converge to a finite value, which may be made arbitrarily small, because $|\Delta_j| < 1$.

parameters in equation 11 are analogous to their real counterparts, although some are a combination of the relative price level and expected inflation rate parameters.[17] However, the inflation factor has an important analytical difference to the real factors discussed in section 3.2, because the mean-reversion coefficient κ_1 is much smaller than for the real factors. Indeed, empirical estimates of κ_1 from Berardi and Esposito (1999) and Brown and Schaefer (1994) are distributed above and below zero, and none are statistically different from zero.[18]

The expected path of the inflation rate with zero mean-reversion may be calculated directly from the expectation of the equation 11 with $\kappa_1 = 0$, i.e $E_t[ds_1(t+m)] = 0$. This is a trivial ordinary differential equation with solution $s_1(t+m) = A_1$, and the boundary condition at $m = 0$ is $s_1(t) = A_1$. The inflation component of the expected short rate is therefore a constant by maturity, with that constant being the current inflation rate, i.e $s_1(t+m) = s_1(t)$.

3.4   The economic interpretation of the VAO model

The results from sections 3.2 and 3.3 show that the generic N-mode VAO model is a natural $(N-1)$-order approximation to the BE model, i.e the dimensionality of the BE model is reduced from J state variables to N factors. The

---

[17]Specifically, $s_1(t) = \pi(t) - \sigma_p^2$; $\kappa_1 = \kappa_\pi$; $\theta_1 = \theta_\pi - \sigma_p^2$; and $\sigma_1 = \sigma_\pi$, where $\pi(t)$ is the expected inflation rate, σ_p^2 is the variance of relative changes in the price level, κ_π is the mean-reversion coefficient for the expected inflation rate, θ_π is the long-term expected inflation rate, and σ_π is the standard deviation of the expected inflation rate.

[18]As noted by Berardi and Esposito (1999), this is consistent with the Fisher hypothesis that changes in nominal long-maturity rates are determined almost exclusively by changes to the expected inflation rate. It is also consistent with the general macroeconomic notion of "sticky prices", or inflation persistence. Note that κ_1 might be non-zero in a regime where monetary policy is directed toward inflation targeting; this proposition, and the implications for the VAO model if κ_1 is non-zero, remain to be investigated in future work.

three-mode VAO model specified in this article is therefore a second-order approximation to the BE model. Four specific observations about the economic interpretation of the VAO model coefficients and parameters may now be advanced.

The first observation is the precise relationship $\beta_1(t) = s_1(t) + \sum_{j=2}^{J} \theta_j$. Hence, $\beta_1(t)$ is the BE long-run equilibrium nominal interest rate partitioned as the BE inflation rate and the BE real long-run equilibrium interest rate (which is synonomous with the real long-run growth rate). A strict interpretation would have the BE real long-run equilibrium interest rate constant, and stochastic changes to $\beta_1(t)$ should reflect unanticipated changes to $s_1(t)$, i.e simultaneous and equal "shocks" to current inflation and expected inflation. However, these relationships are unlikely to be perfectly realised in practice, for several reasons: (1) the real long-run interest rate might change over time; (2) expected inflation could differ materially from current inflation, especially in a transition period (e.g an episode of disinflation); and (3) term premium effects, not considered in the risk-neutral BE model, might emerge at times. Empirical estimates of $\beta_1(t)$ will reflect all of these components as realised in the yield curve data, while the measure of current inflation will not. However, if each component cycles over time and any structural changes are appropriately accounted for, then differences between current inflation and $\beta_1(t)$ should remain stationary; i.e current inflation and $\beta_1(t)$ should be cointegrated with vector (1,-1). This proposition is investigated in section 7.

The second observation is that the levels of the remaining linear coefficients $\beta_n(t)$ reflect the values of the (time-varying) BE real state variable vector

19

$\mathbf{s}\left(t\right)$. In particular, when $m = 0$ there is a precise relationship $\sum_{j=2}^{J} s_j\left(t\right) = -\sum_{n=2}^{N} \beta_n\left(t\right)$, because all OLP modes equal -1 at a maturity of zero. Hence, $-\sum_{n=2}^{N} \beta_n\left(t\right)$ is the BE current real interest rate, which should be synonomous with the current state of the real economy. Stochastic changes to $-\sum_{n=2}^{N} \beta_n\left(t\right)$ should reflect unanticipated changes to $\sum_{j=2}^{J} s_j\left(t\right)$, or "shocks" to the state of the real economy. Once again, this is a strict interpretation, and the empirical relationships in practice remain to be investigated in future work.

The third observation is that the parameter ϕ in the VAO model should be constant, and it may be interpreted as a central measure of the mean-reversion coefficients of the real state variable processes in the BE model, i.e $\phi =$ central(κ_j). Hence, "shocks" to the real economy should persist with an average decay rate of ϕ (i.e a half-life of $\ln\left(2\right)/\phi$).

The fourth observation is that the empirical significance of higher-order modes in the VAO model should indicate the relative distribution of Δ_j, i.e the magnitudes of the mean-reversion coefficients for the real state variables κ_j relative to central(κ_j). If higher-order modes in the VAO model quickly become empirically insignificant, this would suggest that the magnitudes of κ_j are generally similar.[19]

Finally, it is worth noting that reliable empirical relationships between yield curve data and macroeconomic data have been successfully identified in the exisiting literature, such as Estrella and Mishkin (1997), Estrella and Mishkin (1998), Ang and Piazzesi (2003), and Dewachter and Lyrio (2003). The VAO

---

[19]The empirical success of three-mode OLP models in many different markets suggests this is generally the case, but it would be worthwhile specifically investigating the empirical significance of higher-order modes in the VAO model in future work.

model in conjunction with the economic interpretations of this section may offer a less complex avenue for investigating such relationships, and the empirical work in section 7 touches on this. However, a more complete investigation of this potential, such as the out-of-sample forecasting of inflation and GDP data, remains for future work.

4 Description of the data used for the empirical work

To illustrate the general applicability of the VAO model, the empirical work in sections 5 to 7 addresses yield curve fitting (a financial application), yield curve forecasting (a financial/economic application), and the predicted relationship between the Level coefficient and CPI inflation (an economic application).

The data used are obtained from the online Federal Reserve Bank of St Louis economic database (FRED). The interest rate data are monthly averages (of business days) of the federal funds rate (FF), and the yields-to-maturity (on a semi-annual basis) of the 1-year, 3-year, 5-year, 10-year and 20-year constant maturity bonds (GS1, GS3, GS5, GS10, and GS20, respectively). The sample period is July 1954 (the first month for which the FF data is available) to January 2003. However, the GS20 data has a gap from January 1987 to September 1993, and so the monthly averages (of business days) of the generic 30-year bond yield from Bloomberg are used from January 1987 for the remainder of the sample, with an assumed 30-year constant maturity. The slope measure used in the empirical analysis is GS10 less FF, being the spread between the longest and shortest maturity rates available for the entire data period. Figure

3 plots the FF and GS10 data, and figure 4 plots the FF/GS10 slope measure.

[Figure 3 here]

[Figure 4 here]

The VAO model $\boldsymbol{\beta}(t)$ coefficients are used directly as data in some of the empirical analysis. The methods used to estimate $\boldsymbol{\beta}(t)$ from market-quoted interest rate data are documented in the existing literature, but the essential details are contained in Appendix C for completeness. In summary, the estimation process fits a zero-coupon structure to the observed data for the yield curve at each point in time. A time series of data for the yield curve results in a time-series of $\boldsymbol{\beta}(t)$ coefficients associated with an estimated volatility vector $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$. A fixed value of ϕ is also required for the VAO model, and the value of $\phi = 1$ was chosen before undertaking any of the empirical analysis, mainly to avoid any hint of "data mining" in the forecasting application of section 6. However, the alternative values of $\phi = 0.73$, as used in Diebold and Li (2002), and $\phi = 1.27$ were also trialled to establish that the results presented are not critically dependent on the choice of $\phi = 1$.

The CPI inflation series used in section 7 and plotted in figure 11 is an annual measure of "core inflation". This is constructed using the 12 month change in the logarithm of the CPI for all urban consumers to December 1958, the 12 month change in the the logarithm of the CPI ex-food and energy thereafter (the latter index first became available in January 1957).[20]

The four monetary policy regimes noted in the empirical analysis are as

---

[20]Hence, this measure is essentially a 12-month moving-average of monthly inflation. Centering the moving-average or using the monthly series itself makes little difference to the results noted in section 7.

specified in Walsh (1998) and Obstfeld and Rogoff (1999). Calendar years are assumed in all cases, except for the start of the non-borrowed reserves target regime that was specified to an exact month. The regimes are Bretton Woods / gold price target (start of sample to 1971), federal funds rate target (1972 to September 1979), non-borrowed reserves target (October 1979 to 1981, and borrowed reserves / federal funds rate target (1982 to end of sample). Note that these regimes were typically associated with the prevailing economic enviroment at the time,[21] and so any sub-sample results should not necessarily be attributed strictly to the operating regime.

5 The application of the VAO model to the full sample of yield curve data

Following the outline in Appendix C, the volatility vector $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ is first estimated using the full sample of data. That estimate of $\boldsymbol{\sigma}$ is then used to estimate the time-series of $\boldsymbol{\beta}(t)$ coefficients, i.e a value of $\boldsymbol{\beta}(t)$ associated with each observation of yield curve data. The yield residuals from each fitting of yield curve data are used to ascertain the goodness-of-fit of the VAO model, and the $\boldsymbol{\beta}(t)$ coefficients are analysed for their in-sample time-series properties.

---

[21] For example, the withdrawal from Bretton Woods / gold price targeting was partly associated with rising US inflation from the late 1960s, as noted in Obstfeld and Rogoff (1999). The non-borrowed reserves target regime was associated with the disinflation process beginning in the late 1970s, as noted in Walsh (1998).

5.1 The goodness-of-fit of the VAO model

Figure 5 shows yield curve data for the month of September 1992 fitted with the VAO model. This example is actually one of the poorer fits in the sample, but is selected because the positive Slope and negative Bow reflected in the estimated coefficients is visually apparent in the yield curve data, thereby illustrating the intuition behind the estimated cross-sectional coefficients. Figure 6 summarises the cross-sectional goodness-of-fit of the VAO model through time, and the wide variation over the sample is readily apparent. Table 1 summarises the goodness-of-fit of the VAO model using $\phi = 1$, and compares this to VAO models with alternative values of ϕ, and to the VAO model using a five-year centred moving-average window of volatility as discussed in section 5.2. The differences in the average goodness-of-fit to amount to only several basis points in most cases.[22] The OLP model has often been applied to yield curve fitting, and so the results for the OLP model with $\phi = 1$ are included in table 1 as a benchmark. The OLP model fits the cross-sectional data only marginally better than the equivalent VAO model, which indicates that the constraints required to make the VAO model intertemporally consistent do not involve a material cost on the cross-sectional fitting properties relative to the OLP model.

---

[22]The insensitivity of goodness-of-fit to the choice of ϕ in OLP models is already noted in Nelson and Siegel (1987), Barrett et al. (1995), Diebold and Li (2002). However, note from table 1 that increasing the value of ϕ results in a better cross-sectional fit for short-maturity yields to the detriment of long-maturity yields, and vice-versa. This is to be expected; the faster exponential decay by maturity associated with a higher value of ϕ increases the flexibility to fit short-maturity yields, but that faster decay to zero also effectively leaves only the Level mode to fit longer-maturity yields.

5.2 The time series properties of the VAO model coefficients

Figure 7 plots the time series of Slope and Bow coefficients estimated over the full sample. The time series for the Level coefficient is plotted in figure 11. As examples of the intuition behind the time-series of cross-sectional coefficients, note the sharp steepening of the yield curve (i.e an increase in the Slope coefficient) from 1989 to 1992, and the general decline in yields (i.e a decrease in the Level coefficient) from the 1981 peak.

[Figure 7 here]

The derivation of the VAO model assumes independent and constant Gaussian distributions for changes to the β_n coefficients, and these assumptions can now be checked. Figure 8 shows that the null hypothesis of a constant volatility (i.e the standard deviation of changes) in the Level coefficient over the entire sample can be rejected, and similar results (and patterns of volatility) are found for the Slope and Bow coefficients. These results do not invalidate the application of the VAO model; unanticipated switching of volatility to a different constant (as could be argued in this application, given that different levels of volatility appear to be coincident with different monetary policy regimes) is still consistent with the VAO model assumptions. However, to ensure the results are not sensitive to the assumption of constant volatility, the estimation of the VAO model coefficients over the full sample of data is repeated using a five-year centred moving-average window of volatility, and changes to the results are immaterial (as shown in tables 1, 2 and 6).

Regarding the independence of changes to the β_n coefficients, figure 9 shows

25

that the assumption of zero correlation between changes in Level and Bow seems reasonable (and the results between Level and Slope are similar), but the correlations between monthly changes in Slope and Bow are significantly negative over the entire sample. However, this will not have a material effect in empirical applications because the Slope and Bow volatility adjustment terms are very small relative to that for Level volatility (as shown in figure 2).[23]

Given the evidence for heteroskedasticity noted above (i.e the non-constant variances of $\Delta\beta_n$), Phillips-Perron unit root tests are applied to the VAO model coefficients to determine their time-series properties, and the results are summarised in table 2.[24] The results are consistent with the predictions in section 2.4; i.e a unit root process cannot be rejected for the Level series, but is rejected for the Slope and Bow series.

6 Forecasting the yield curve with the VAO model

Section 6.1 investigates the yield curve forecasting potential of the risk-neutral of the VAO model, which is equation 9 with $\boldsymbol{\mu} = \{0, 0, 0\}'$. Section 6.2 illustrates the application of the VAO model to yield curve forecasting using an estimated term premium function.

---

[23] In any case, the zero covariance assumption used in the derivation of the VAO model is a convenient rather than critical assumption; eigenvector analysis could be used to obtain new modes as a linear combination of the original OLP modes, where changes in the coefficients of those new modes would have zero covariance.

[24] For this and the tests in section 7, the Bartlett window, as specified in Newey and West (1987), is used to ensure that the calculated variance is positive. The bandwidth of the window is selected using the method outlined in Newey and West (1994). The levels of significance are gauged using the critical statistics from Hamilton (1994) pp. 763-756.

6.1 Forecasting with the risk-neutral version of the VAO model

The steps for forecasting the yield curve as at time t for horizon τ (i.e a forecast for time $t+\tau$) are as follows: (1) the estimate of the constant volatility vector $\boldsymbol{\sigma}$ as at time t, denoted $\boldsymbol{\sigma}(t)$, is calculated from the component data using $\sigma_n^2(t) = \frac{12}{t-1}\sum_{i=0}^{t-1}[\Delta\beta_n(i)]^2$ (where i is a dummy summation variable for historical time, and the scalar 12 annualises the monthly variances);[25] (2) the coefficients $\boldsymbol{\beta}(t)$ are estimated using $\boldsymbol{\sigma}(t)$ and the observation of yield curve data for time t; (3) the forecast VAO model coefficients for time $t + \tau$, i.e $E_t[\boldsymbol{\beta}(t+\tau)]$, are calculated using equation 9 with $\boldsymbol{\mu} = \{0,0,0\}'$; and (4) the forecast yields for the required maturities at time $t+\tau$ are re-constructed using $E_t[\boldsymbol{\beta}(t+\tau)]$ and the value of $\boldsymbol{\sigma}(t)$ applied to the forecast date (which is consistent with the assumption of constant volatility). The forecast bond yields are re-constructed by assuming they apply to a par bond, i.e with a principal set equal to 1 for the given maturity, the semi-annual coupons that give a settlement price of -1 are calculated, and the yield forecast is twice that coupon rate. The forecast FF/GS10 spread is calculated as the GS10 forecast less the FF forecast.

The random walk is the typical näive benchmark used to assess relative forecasting performance.[26] Under the random walk process the yields at time $t + \tau$ are identical to those prevailing at time t.

---

[25] This is the most näive method of recursively updating the estimate of $\boldsymbol{\sigma}$ from the historical data available at the time the forecast is made, and was chosen from the outset to avoid any hint of "data mining" a favourable weighting structure or moving-average window for the estimate of $\boldsymbol{\sigma}$.

[26] The OLP model has also been successfully used by Diebold and Li (2002) to forecast the US yield curve, and is another potential benchmark. Using the OLP model with the data from this article gave results within a few basis points of the VAO model (the results are not shown due to space limitations, but are available from the author). This suggests that an intertemporally consistent model is not necessarily a critical requirement for practical forecasting applications.

Table 3 contains the root-mean-square (RMS) forecast errors for the VAO model. To save space, only the results for FF (the shortest maturity yield), GS10 (the longest maturity yield available over the full sample), and FF/GS10 spread (the widest spread available for the entire sample) are shown; the results for intermediate maturities and spreads generally fall between these sets of results. The RMS forecast errors broadly show an increase by horizon, as expected because the yield curve is subject to more "new information" from the time of forecast. The magnitudes of the RMS forecast errors in each regime broadly follow the pattern of volatility in figure 8, also as expected because higher yield curve volatility should result in larger forecast errors.

Table 4 compares the RMS forecast errors for the VAO model to the random walk. The statistical significance of each entry is estimated using the Diebold and Mariano (1995) method with the bandwidth set equal to the horizon of the forecast (in months) less 1.[27] Over the whole sample, the VAO model forecasts for FF and FF/GS10 outperform those of the random walk, and the magnitude and significance of the outperformance tends to rise for longer forecast horizons. However, the VAO model forecasts for GS10 consistently underperform the random walk over the full sample for all horizons. The sub-period results offers some insight into the GS10 results; the general outperformance of the VAO model during the Bretton Woods and federal funds rate target regimes is more than counterbalanced by the underperformance during the non-borrowed and

---

[27]This is the proceedure suggested in Diebold and Mariano (1995) and used in Diebold and Li (2002), because it allows for overlapping forecasting errors due to the frequency of the data being greater than the forecast horizon. Note that the small size of the non-borrowed reserves sub-sample means that statistical significance cannot be ascertained using the Diebold and Mariano (1995) test, so no indications are given.

borrowed reserves target regimes. Another interesting aspect during the latter regime is that the significance of the VAO model forecasts for FF moves from an outperformance of the random walk for shorter horizons, to an increasing underperformance for longer horizons.

Further investigation into the poor forecasting performance of the VAO model during the borrowed reserves regime suggested a term premium effect for all of the horizons investigated, and so the forecasting excercise was repeated with an estimated term premium, as discussed in section 6.2.

6.2   Forecasting with a term premium in the VAO model

To illustrate the term premium effect, the forecasting excercise outlined in section 6.1 is repeated for the one-year horizon over the out-of-sample period February 1994 to January 2003. Estimates of the one-year term premium and $\boldsymbol{\sigma}$ relevant to this sub-sample are calculated from the historical period October 1986 to January 1994, i.e $\sigma_n^2 = \frac{12}{76} \sum_{i=\text{Oct-86}}^{\text{Jan-94}} [\Delta \beta_n(i)]^2$ and $\boldsymbol{\mu} = \frac{1}{76} \sum_{t=\text{Oct-86}}^{\text{Jan-94}} \{\boldsymbol{\Phi}(\phi, 1) \boldsymbol{\beta}(t-1) - \beta_n(t)\}$. This estimation period is chosen because it spans the first full monetary policy cycle (i.e a trough-to-trough cycle in the federal funds rate, and a full cycle in long–maturity yields) following the re-establishment of price stability and inflation credibility from around the mid-1980s. The estimate of the one-year term premium is $\boldsymbol{\mu} = \{0.23, -1.62, 1.43\}'$ percentage points, and the volatility estimate is $\boldsymbol{\sigma} = \{0.84, 1.40, 1.12\}'$ percentage points.

Table 5 firstly shows that all yield forecasts are biased upwards using the new estimate of $\boldsymbol{\sigma}$ but setting $\boldsymbol{\mu} = \{0, 0, 0\}'$. This illustrates the term premium

29

effect, which leads to higher RMS forecast errors than for the random walk. However, allowing for the estimated term premium leads to an approximately zero bias for all yield forecasts, and the RMS errors for all yield forecasts are lower than for the random walk. Note that the outperformance of the random walk is similar to the out-of-sample results of Duffee (2002) using comparable maturities and horizons (e.g 7.4 basis points for the 10-year rate on a one year horizon, compared to 7 basis points here), but the VAO model is substantially less complex than the "essentially affine" models of Duffee (2002). However, the outperformance is smaller than in Diebold and Li (2002) (e.g 27.4 basis points for the 10-year rate on a one year horizon), which suggests that the more complex generalised autoregressive conditional heteroskedasticity (GARCH) model used in that paper might offer an additional avenue for improving yield curve forecasts.

The shape of the point estimate of the term premium function, i.e $\boldsymbol{\mu}'\mathbf{g}(\phi, m)$, is illustrated in figure 10. As noted in section 2.1, the term premium is a physical realisation of the market pricing for all sources of risk, and so it is not possible to identify why this particular term premium effect has arisen during the borrowed reserves regime. However, the success of the risk-neutral model (i.e with no term premium) for forecasting the yield curve during the Bretton Woods and federal funds target regimes does suggest that the term premium effects are not present in the earlier part of the sample.

7 The relationship between the VAO Level coefficient and CPI inflation

As a simple illustration of the economic interpretation of the VAO model coefficients, this section tests the section 3.4 prediction of cointegration between the VAO level coefficient and CPI inflation. Figure 11 plots CPI inflation and the Level coefficient of the VAO model, and figure 12 plots the Level coefficient less CPI inflation. In the latter figure, a potential structural change is apparent from around the non-borrowed reserves target regime, which coincides with the beginning of the disinflation period as noted in Walsh (1998). The hypothesis of structural change in the Level coefficient less CPI inflation series is supported statistically, i.e regressing that series on a step dummy variable from October 1979 results in a t-statistic of 2.00 (after adjusting for the serial correlation of residuals), which is within the 5 percent level of significance.

Table 6 shows the results of unit root and cointegration tests. Firstly, a unit root process cannot be rejected for CPI inflation (as for the Level coefficient results in table 2). Secondly, the hypothesis that the Level coefficient less CPI inflation series contains a unit root is weakly rejected. Thirdly, after allowing for the estimated structural change from October 1979, the hypothesis of a unit root in the residuals is rejected. Overall, these results indicates that CPI inflation and the Level coefficient are cointegrated with a vector of (1,-1), and with more significance when accounting for an estimated structural change from October 1979.

8   Conclusion

Since its introduction by Nelson and Siegel (1987), the OLP model of the yield curve, in various forms, has proved popular with both researchers and market practioners. The VAO model continues the tradition of the OLP model, in being simple, intuitive, and empirically robust, but makes two important extensions. Firstly, the VAO model is derived using the Heath, Jarrow and Morton (1992) risk-neutral framework as a foundation, which ensures cross-sectional and intertemporal consistency. Secondly, the VAO model parameters and coefficients are shown, via a comparison to the Berardi and Esposito (1999) model of the forward rate curve developed under a generic general equilibrium framework, to have a direct economic interpretation.

Using United States data, the empirical work in this article illustrates the potential of the VAO model in financial and economic applications. For example, the VAO model significantly outperforms the random walk when used to forecast the yield curve. Also, as predicted, CPI inflation is shown to have a cointegrating relationship with the level of the yield curve as measured by the VAO model. In both applications there is evidence of a structural change from around 1979, which coincides with the beginning of the disinflation period during the non-borrowed reserves monetary policy regime, as noted in Walsh (1998).

In summary, researchers and market practioners requiring a convenient and theoretically robust model of the yield curve should find the VAO model a useful tool for a wide variety of applications.

A The generic volatility-adjustment calculation for the VAO model

Define an exponential-polynomial volatility function as:

$$\sigma(s, m) = \sigma \cdot \exp(-\phi m)[\phi m]^a \tag{15}$$

where a (≥ 0) is an integer. Following the approach of HJM, the integral $\int_s^m \sigma_n(s, u) \, du$ is first calculated as:

$$\sigma \cdot \int_s^m \exp(-\phi[m - u])[\phi(m - u)]^a \, du \tag{16a}$$

$$= \ \sigma \cdot \left[-\frac{1}{\phi}\Gamma[1 + a, \phi(m - u)]\right]_s^m \tag{16b}$$

$$= \ \frac{\sigma}{\phi} \cdot (-\Gamma[1 + a, \phi(m - s)] + \Gamma[1 + a, 0]) \tag{16c}$$

where $\Gamma(\cdot)$ is the incomplete Gamma function, and is defined as $\Gamma(a, z) = \int_z^\infty w^{a-1} \exp(-w) \, dw$.[28] Note that $\Gamma(1 + a, 0) = \int_0^\infty z^a \exp(-z) \, dz = \Gamma(1 + a) = a!$, the factorial definition, and these expressions are used interchangeably below. Substituting the result from equation 16c into $\sigma_n(s, m)\left[\int_s^m \sigma_n(s, u) \, du\right] ds$, the drift expression component from equation 1, gives:

---

[28] See, for example, Wolfram (1996) p. 740.

$$\frac{\sigma^2}{\phi} \cdot \int_0^m (\phi \, [m-s])^a \exp(-\phi \, [m-s])$$

$$\times (-\Gamma \, [1+a, \phi \, (m-s)] + a!) \, ds \qquad (17a)$$

$$= \frac{\sigma^2}{2\phi^2} \left[2a! \Gamma \, [1+a, \phi \, (m-s)] - (\Gamma \, [1+a, \phi \, (m-s)])^2 \right]_0^m \qquad (17b)$$

$$= \frac{\sigma^2}{2\phi^2} \left[2a! - (a!)^2 - 2a! \cdot \Gamma \, [1+a, \phi m] + (\Gamma \, [1+a, \phi m])^2 \right] \qquad (17c)$$

$$= \frac{\sigma^2}{2\phi^2} (a! - \Gamma \, [1+a, \phi m])^2 \qquad (17d)$$

To calculate $h_n(\phi, m)$ for $n > 1$, the generic OLP volatility function is written as a summation of exponential-polynomial terms, and the corresponding results from equation 17d are applied. That is, $\sigma_n(m) = \sigma_n \cdot g_n(\phi, m) = \sigma_n \cdot \exp(-\phi m) \cdot \sum_{k=0}^{n-2} \frac{(-1)^k (n-2)! (2\phi m)^k}{(k!)^2 (n-2-k)!} = \sigma_n \cdot \sum_{k=0}^{n-2} \frac{(-2)^k (n-2)!}{(k!)^2 (n-2-k)!} \exp(-\phi m) \, [\phi m]^k$, and therefore:

$$h_n(\phi, m) = \frac{1}{2\phi^2} \cdot \sum_{k=0}^{n-2} \frac{(-2)^k (n-2)!}{(k!)^2 (n-2-k)!} \cdot (k! - \Gamma \, [1+k, \phi m])^2 \qquad (18)$$

B   The time evolution of the expected path of the short rate in the HJM model

Firstly, rewrite equation 1 for an arbitrary time t, and then apply the expectations operator at time t to obtain the result:

$$f(t, m) = E_t \, [r \, (t+m)] - \sum_{n=1}^N \int_t^{t+m} \alpha_n \, (s, m) \, ds \qquad (19)$$

Introduce a time-increment τ to denote a finite evolution in time from time t to $t+\tau$, and use equation 19 to specify the following two relationships at times t and $t + \tau$:

$$f(t, \tau + m) = E_t[r(t + \tau + m)] - \sum_{n=1}^{N} \int_t^{t+\tau+m} \alpha_n(s, m)\, ds \qquad (20a)$$

$$f(t + \tau, m) = E_{t+\tau}[r(t + \tau + m)] - \sum_{n=1}^{N} \int_{t+\tau}^{t+\tau+m} \alpha_n(s, m)\, ds \qquad (20b)$$

Equation 4 in the HJM paper gives the relationship between the forward rate curve at times t and $t + \tau$ as:[29]

$$f(t + \tau, m) = f(t, \tau + m) + \sum_{n=1}^{N} \int_t^{t+\tau} \alpha_n(s, m)\, ds + \sum_{n=1}^{N} \int_t^{t+\tau} \sigma_n(s, m)\, d\tilde{W}_n(s)$$

$$(21)$$

Substituting equations 20a and 20b into equation 21 gives the equality:

$$E_{t+\tau}[r(t + \tau + m)] - \sum_{n=1}^{N} \int_{t+\tau}^{t+\tau+m} \alpha_n(s, m)\, ds \qquad (22a)$$

$$= E_t[r(t + \tau + m)] - \sum_{n=1}^{N} \int_t^{t+\tau+m} \alpha_n(s, m)\, ds$$

$$+ \sum_{n=1}^{N} \int_t^{t+\tau} \alpha_n(s, m)\, ds + \sum_{n=1}^{N} \int_t^{t+\tau} \sigma_n(s, m)\, d\tilde{W}_n(s) \qquad (22b)$$

---

[29]Note that m on the left-hand side expression of the forward rate curve and $\tau + m$ on the right-hand side of the forward rate curve refer to the same future point in time, which is denoted by T in the HJM paper.

Equation 22b contains two identical integrals with different upper limits of integration. These may be combined into a single integral with a new lower limit of integration, i.e:

$$-\sum_{n=1}^{N}\int_{t}^{t+\tau+m}\alpha_n(s,m)\,ds+\sum_{n=1}^{N}\int_{t}^{t+\tau}\alpha_n(s,m)\,ds = -\sum_{n=1}^{N}\int_{t+\tau}^{t+\tau+m}\alpha_n(s,m)\,ds$$

(23)

This shows that the deterministic drift terms in equations 22a and 22b are identical. Removing them from both sides of the equality leaves the result:

$$E_{t+\tau}\left[r\left(t+\tau+m\right)\right] = E_t\left[r\left(t+\tau+m\right)\right] + \sum_{n=1}^{N}\int_{t}^{t+\tau}\sigma_n(s,m)\,d\tilde{W}_n(s) \quad (24)$$

The individual integrals in the final summation term do not have closed form solutions. However, the results noted in Ross (1997) pp. 541-542 show that $E_t\left[\int_t^{t+\tau}\sigma_n(s,m)\,d\tilde{W}_n(s)\right]=0$. Also, each integral $\int_t^{t+\tau}\sigma_n(s,m)\,d\tilde{W}_n(s)$ is a "summation" of $\sigma_n(s,m)$ increments, as specified in the initial set-up of equation 1. Therefore $\int_t^{t+\tau}\sigma_n(s,m)\,d\tilde{W}_n(s)$ will be of the functional form $\varepsilon(\tau)\cdot\sigma_n(s,m)$, where $\varepsilon(\tau)$ has an expected value of zero (but will not necessarily be Gaussian).

C The empirical application of the VAO model to market-quoted interest rate data

At each point in time, the VAO model coefficients β_n are estimated using the characteristics and market-quoted data for the fixed interest instruments that represent the yield curve at that time. Those instruments are typically coupon-bearing, and so an allowance for multiple cashflows (each with a different zero-coupon discount rate corresponding to the timing of the cashflow) is required, i.e:[30]

$$\text{Minimise} \quad : \quad \sum_{k=1}^{K} (w_k \cdot \varepsilon_k)^2 \tag{25a}$$

$$\text{where} \quad : \quad \varepsilon_k = \sum_{j=1}^{J[k]} a_{jk} \cdot \exp\left[-m_{jk} \cdot R\left(m_{jk}\right)\right] \tag{25b}$$

$$\text{and} \quad : \quad R\left(m\right) = \sum_{n=1}^{N} \beta_n \cdot s_n(\phi, m) - \sum_{n=1}^{N} \sigma_n^2 \cdot u_n(\phi, m) \tag{25c}$$

where:

- K is the number of fixed interest instruments used to define the yield curve;

- w_k is a weighting factor, which is set to the inverse of the "basis point value" (i.e the price change of the instrument for a yield change of a single

---

[30] This is the most widely used approach for estimating OLP model coefficients from market-quoted data, and is outlined in Söderlind and Svensson (1997) and the articles in the Bank for International Settlements (1999). Zero-coupon interest rate data, as used in Diebold and Li (2002), can also be used within this set-up by specifying just two cashflows for each instrument.

basis point) in this article to obtain a minimisation of yield residuals (price residual minimisation is acheived by using equal weights);

- $J[k]$ is the number of cashflows for instrument k;

- a_{jk} is the magnitude of the cashflow j for instrument k (defined to be negative for the settlement price, and positive for all cashflows beyond settlement);

- m_{jk} is the maturity of the cashflow j of instrument k; and

- $R(m_{jk})$ is the zero-coupon interest rate.

Expression 25 is estimated using the the Newton-Raphson technique. The zero-coupon interest rates in equation 25c are calculated from the VAO model forward rate curve by integrating the corresponding forward rate components. That is, $R(m) = \frac{1}{m} \int_0^m f(m) dm$, and so $s_n(\phi, m) = \frac{1}{m} \int_0^m g_n(\phi, m) dm$, and $u_n(\phi, m) = \frac{1}{m} \int_0^m h_n(\phi, m) dm$. The relevant results for $s_n(\phi, m)$ and $u_n(\phi, m)$ in the three-mode VAO model are, respectively:

$$s_1(\phi, m) = 1 \tag{26a}$$

$$s_2(\phi, m) = \frac{1}{\phi m} [\exp(-\phi m) - 1] \tag{26b}$$

$$s_2(\phi, m) = -\frac{1}{\phi m} [2\phi m \exp(-\phi m) + \exp(-\phi m) - 1] \tag{26c}$$

$$u_1(\phi, m) \;=\; \frac{1}{6}m^2 \tag{27a}$$

$$u_2(\phi, m) \;=\; \frac{1}{4\phi^3 m}\left[4\exp\left(-\phi m\right) - 3 + 2\phi m - \exp\left(-2\phi m\right)\right] \tag{27b}$$

$$u_3(\phi, m) \;=\; \frac{1}{2\phi^3 m}\left[\begin{array}{c} 6\exp\left(-\phi m\right) - 3\phi m\exp\left(-2\phi m\right) \\[2mm] +4\phi m\exp\left(-\phi m\right) - 4 + \phi m \\[2mm] -\phi^2 m^2\exp\left(-2\phi m\right) - 2\exp\left(-2\phi m\right) \end{array}\right] \tag{27c}$$

In this article, the data used are monthly averages of interpolated constant-maturity yields on a semi-annual basis, and so the precise cashflows are not available (i.e the data is essentially an average of bond yields across maturity and time). Hence, the yield is assumed to correspond to a par bond for the specified maturity, so the cashflows are a settlement price of -1, a principal of 1 for the given maturity, and semi-annual coupons between settlement and maturity equal to half the yield.

Note that ϕ and the volatility coefficients σ_n in equation 25c are pre-specified parameters. ϕ may be calibrated with regard to historical data, although empirical results are generally insensitive to the exact choice, as noted in section 5.1. σ_n may be calculated from historical data over a suitable period by annualising the usual definition of variance, i.e $\sigma_n^2 = \frac{12}{t}\sum_{i=1}^{t}[\Delta\beta_n(i)]^2$ is used for the monthly data used in this article.[31] An initial estimate of the $\boldsymbol{\beta}(t)$ coefficients to use for this volatility calculation can be obtained by firstly assuming a zero volatility, i.e $\sigma_n^2 = 0$, in equation 25. The estimation of equation 25 is then

---

[31]σ_n could also potentially be calibrated from data for options on interest rate instruments observed at the same time as the yield curve data, if such data are available.

repeated using the initial estimates of σ_n^2 to calculate the $\boldsymbol{\beta}(t)$ coefficients.[32]

References

Ang, A. and Piazzesi, M. (2003), 'A no-arbitrage vector autoregression of term structure dynamics with latent variables', *Journal of Monetary Economics* **50**, 745–787.

Bank for International Settlements (1999), *Zero-coupon yield curves: technical documentation*, Bank for International Settlements, Switzerland.

Barrett, W., Gosnell, T. and Heuson, A. (1995), 'Yield curve shifts and the selection of immunization strategies', *Journal of Portfolio Management* **Fall**, 53–64.

Berardi, A. and Esposito, M. (1999), 'A base model for multifactor specifications of the term structure', *Banca Monte dei Paschi di Siena* **28(2)**, 145–170.

Björk, T. and Christensen, B. (1999), 'Interest rate dynamics and consistent forward rate curves', *Mathematical Finance* **9(4)**, 323–348.

Bliss, R. (1997), 'Testing term structure estimation methods', *Advances in Futures and Options Research* **9**, 197–231.

Brandt, M. and Yaron, A. (2002), 'Time-consistent no-arbitrage models of the term structure', *Working paper, Wharton School,University of Pennsylvania* .

---

[32]This process could be iterated to covergence, but the changes are negligible after just the two steps described in the text.

Brooks, R. and Yong Yan, D. (1999), 'London inter-bank offer rate (LIBOR) versus Treasury rate: evidence from the parsimonious term structure model', *Journal of Fixed Income* **June**, 71–83.

Brown, R. and Schaefer, S. (1994), 'Interest rate volatility and the shape of the term structure', *Philosophical Transactions of the Royal Society* **A, 347**, 563–576.

Courant, R. and Hilbert, D. (1953), *Methods of Mathematical Physics, First English Edition*, Interscience Publishers Inc., New York.

Cox, J., Ingersoll, J. and Ross, S. (1985*a*), 'An intertemporal general equilibrium model of asset prices', *Econometrica* **53**, 363–384.

Cox, J., Ingersoll, J. and Ross, S. (1985*b*), 'A theory of the term structure of interest rates', *Econometrica* **53**, 385–407.

Dahlquist, M. and Svensson, L. (1996), 'Estimating the term structure of interest rates for monetary policy analysis', *Scandinavian Journal of Economics* **98(2)**, 163–183.

Dai, Q. and Singleton, K. (2000), 'Specification analysis of affine term structure models', *Journal of Finance* **55(5)**, 1943–1978.

de La Grandville, O. (2001), *Bond Pricing and Portfolio Analysis: Protecting Investors in the Long Run*, The MIT Press.

Dewachter, H. and Lyrio, M. (2003), 'Macro factors and the term structure of interest rates', *Working paper, Catholic University of Leuven* .

Diebold, F. and Li, C. (2002), 'Forecasting the term structure of government bond yields', *Working paper, University of Pennsylvania* .

Diebold, F. and Mariano, R. (1995), 'Comparing predictive accuracy', *Journal of Business and Economic Statistics* **13(3)**, 253–263.

Duffee, G. (2002), 'Term premia and interest rate forecasts in affine models', *Journal of Finance* **57(1)**, 405–443.

Duffie, D. and Kan, R. (1996), 'A yield-factor model of interest rates', *Mathematical Finance* **6(4)**, 379–406.

Estrella, A. and Mishkin, F. (1997), 'The predictive power of the term structure of interest rates in Europe and the United States: implications for the European Central Bank', **41(7)**, 1375–1401.

Estrella, A. and Mishkin, F. (1998), 'Predicting U.S. recessions: financial variables as leading indicators', **80(1)**, 45–61.

Fang, V. and Muljono, R. (2003), 'An empirical analysis of the Australian dollar swap spreads', *Pacific-Basin Finance Journal* **11**, 153–173.

Fergusson, R. and Raymar, S. (1998), 'A comparative analysis of several popular term structure estimation models', *Journal of Fixed Income* **March**, 17–32.

Filopović, D. (1999*a*), 'Exponential-polynomial families and the term structure of interest rates', *Working paper, ETH Zurich* .

Filopović, D. (1999b), 'A note on the Nelson-Siegal family', *Mathematical Finance* **9(4)**, 349–359.

Hamilton, J. (1994), *Time Series Analysis*, Princeton University Press.

Heath, D., Jarrow, R. and Morton, A. (1992), 'Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation', *Econometrica* **60(1)**, 77–106.

Ho, T. and Lee, S. (1986), 'Term structure movements and pricing interest rate contingent claims', *Journal of Finance* **41**, 1011–1029.

Hull, J. (2000), *Options, Futures and Other Derivitives, Fourth Edition*, Prentice Hall International.

Hull, J. and White, A. (1990), 'Pricing interest rate derivative securities', *Review of Financial Studies* **3(4)**, 573–592.

Hunt, B. (1995), 'Fitting parsimonious yield curve models to Australian coupon bond data', *Working paper, School of Finance and Economics, University of Technology Sydney* **51**.

Ioannides, M. (2003), 'A comparison of yield curve estimation techniques using UK data', *Journal of Banking and Finance* **27**, 1–26.

Jordan, J. and Mansi, S. (2003), 'Term structure estimation from on-the-run Treasuries', *Journal of Banking and Finance* **27**, 1487–1509.

Kacala, V. (1993), 'Zero coupon curves Part 3: Fair bond yields', *Research paper, Buttle Wilson* .

Krippner, L. (2002), 'Modelling the yield curve with orthonormalised Laguerre polynomials: a consistent cross-sectional and inter-temporal approach', *Working paper, University of Waikato* .

Mansi, S. and Phillips, J. (2001), 'Modeling the term structure from the on-the-run Treasury yield curve', *Journal of Financial Research* **24(4)**, 545–564.

Monetary Authority of Singapore (1999), 'Extracting market expectations of future interest rates from the yield curve: an application using Singapore interbank and interest rate swap data', *Occasional paper 17, Economics Department, Monetary Authority of Singapore* .

Nelson, C. and Siegel, A. (1987), 'Parsimonious modelling of yield curves', *Journal of Business* **October**, 473–489.

Newey, W. and West, K. (1987), 'A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix', *Econometrica* **55**, 703–708.

Newey, W. and West, K. (1994), 'Automatic lag selection in covariance matrix estimation', *The Review of Economic Studies* **61**, 631–653.

Obstfeld, M. and Rogoff, K. (1999), *Foundations of International Macroeconomics*, The MIT Press.

Rainville, E. and Bedient, P. (1981), *Elementary Differential Equations, Sixth Edition*, Collier Macmillan Publishers.

Ross, S. (1997), *Introduction to Probability Models, Sixth Edition*, Academic Press Limited.

Schich, S. (1997), 'Estimating the German term structure', *Discussion Paper, Deutsche Bundesbank, Economic Research Group* **4**.

Schmid, B. and Kalemanova, A. (2002), 'Applying a three-factor defaultable term structure model to the pricing of credit default options', *International Review of Financial Analysis* **11**, 139–158.

Seppala, J. and Viertio, P. (1996), 'The term structure of interest rates: estimation and interpretation', *Discussion paper, Bank of Finland* **19**.

Söderlind, P. and Svensson, L. (1997), 'New techniques to extract market expectations from financial instruments', *Journal of Monetary Economics* **40**, 383–429.

Soto, G. (2001), 'Immunization derived from a polynomial duration vector in the Spanish bond market', *Journal of Banking and Finance* **25**, 1037–1057.

Svensson, L. (1994), 'Estimating and interpreting forward interest rates: Sweden 1992-4', *Discussion paper, Centre for Economic Policy Research* **1051**.

Vasicek, O. (1977), 'An equilibrium characterisation of the term structure', *Journal of Financial Economics* **5**, 177–188.

Walsh, C. E. (1998), *Monetary theory and policy*, The MIT Press.

Wolfram, S. (1996), *The Mathematica Book, Third Edition*, Cambridge University Press.

Model RMS yield residual	FF	GS1	GS3	GS5	GS10	Cross-sect. 1	GS20	GT30	Cross-sect. 2
VAO(ϕ=1) model	4.7	13.9	10.9	11.8	14.9	10.0	12.4	22.6	10.9
relative to ϕ=0.73	-3.0	-4.7	-1.6	1.7	0.8	-0.7	2.0	8.2	0.1
relative to ϕ=1.27	0.9	-0.4	0.4	-4.0	-0.1	-0.7	-2.7	-9.0	-1.5
rel. to ϕ=1 m/a vol.	0.2	0.7	0.2	-0.3	-0.5	0.3	-0.1	-0.5	0.3
OLP(ϕ=1) model	4.7	14.2	11.1	9.8	8.1	7.7	12.4	9.9	7.9

Table 1: Yield residual analysis for the $\phi = 1$ VAO model, and a relative comparison to alternative VAO models; a negative entry (shaded) indicates that the $\phi = 1$ VAO model has a superior goodness-of-fit to the alternative. Cross-section 1 is the average of the time-series of cross-sectional RMS yield residuals for FF to GS10 (the time series plotted in figure 6), and Cross-section 2 adds either the GS20 and GS30 yield residual to the RMS calculation. The results for the $\phi = 1$ OLP model are shown as a benchmark.

Model	Estimated parameters	Level	Slope	Bow
Constant	AR1 coefficient	0.995	0.967	0.887
volatility	Phillips-Perron statistic	-1.75	-3.23 **	-5.81 ***
VAO	Window bandwidth	18	17	17
5-year m/a	AR1 coefficient	0.995	0.969	0.906
volatility	Phillips-Perron statistic	-1.69	-3.10 **	-5.13 ***
VAO	Window bandwidth	18	18	17

Table 2: Phillips-Perron unit root tests on the time series of the Level coefficient plotted in figure 11, and the Slope and Bow coefficients plotted in figure 7.
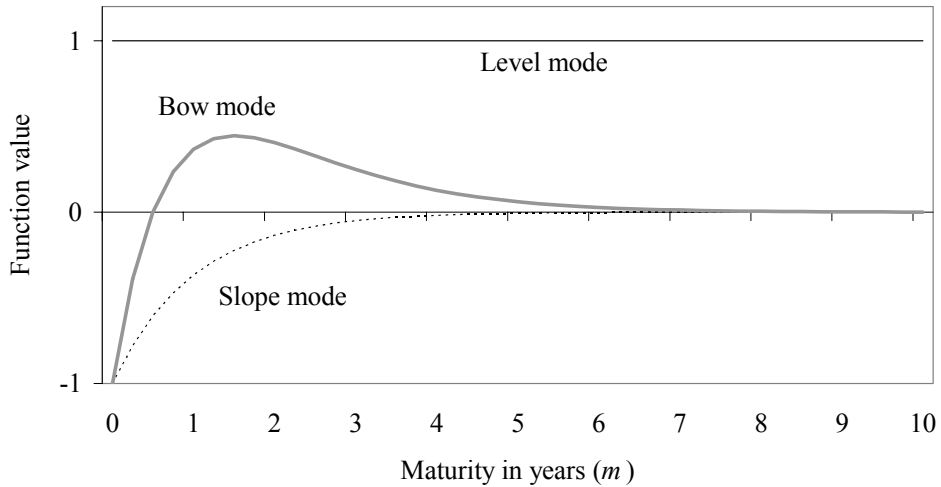

Figure 1: The first three OLP modes with $\phi = 1$.

Forecast horizon (years)	Yield or spread forecast	Monetary policy regime				
		Full sample	Bretton-Woods / gold price target	Federal funds rate target	Non-borrowed reserves target	Borrowed reserves / federal funds rate target
0.25	FF	119	66	134	410	67
	GS10	62	31	34	144	72
	FF/GS10	100	53	121	316	71
0.5	FF	168	108	208	507	111
	GS10	88	46	47	160	109
	FF/GS10	129	82	179	365	89
1	FF	221	148	270	669	172
	GS10	129	64	66	272	159
	FF/GS10	152	109	219	429	111
1.5	FF	259	158	298	655	225
	GS10	158	75	87	348	186
	FF/GS10	160	116	230	339	126
2	FF	284	151	279	400	270
	GS10	176	83	96	419	203
	FF/GS10	163	114	209	43	136
3	FF	311	155	162	n/a	334
	GS10	204	105	94	n/a	237
	FF/GS10	169	115	166	n/a	148

Table 3: RMS VAO model forecast errors, by horizon and operating regime.
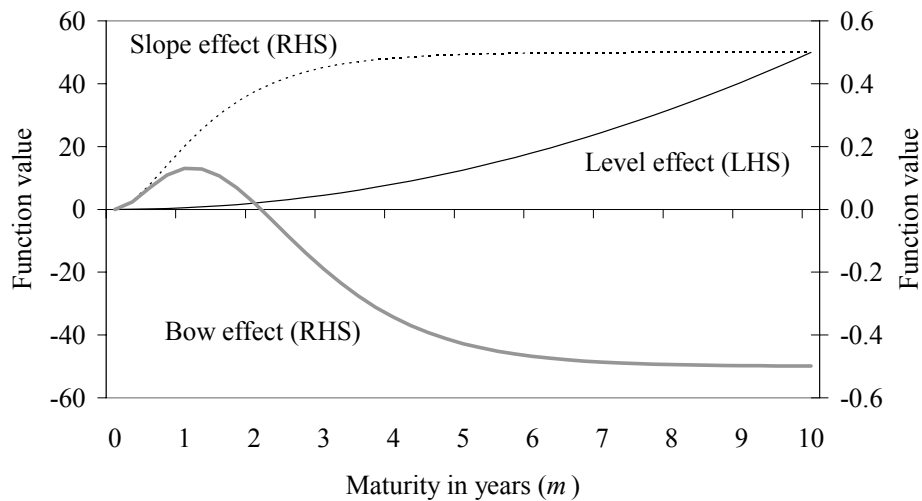
Figure 2: The first three volatility adjustment functions, $h_n(\phi, m)$, with $\phi = 1$.

Forecast horizon (years)	Yield or spread forecast	Monetary policy regime				
		Full sample	Bretton-Woods / gold price target	Federal funds rate target	Non-borrowed reserves target	Borrowed reserves / federal funds rate target
0.25	FF	-2	3	10	-26	-9 *
	GS10	4 ***	-1 ^	0	2	8 ***
	FF/GS10	-5	1	10	-53	0
0.5	FF	0	3	3	16	-13 ^^
	GS10	5 *	-2 ^	-2	1	10 **
	FF/GS10	-7	-1	-4	-21	-10 ^
1	FF	-9	-9	-38 ^	108	-13 ^
	GS10	9 *	-4 *	-5	4	17 *
	FF/GS10	-33 *	-10	-64 ^	43	-35 **
1.5	FF	-26 ^	-28 ^^	-116 ***	284	4
	GS10	11 ^^	-5 *	-11 ^	19	26 **
	FF/GS10	-60 **	-20	-145 **	59	-45 ***
2	FF	-38 ^^	-37 ^^	-163 ***	289	21
	GS10	13 *	-4 ^^	-15 ^^	45	34 ***
	FF/GS10	-75 **	-23	-189 ***	-330	-43 ***
3	FF	-46 ^^	-9	-231 ***	n/a	47 **
	GS10	16 ^^	-4 ^	-16 *	n/a	50 ***
	FF/GS10	-86 **	19 ^	-202 ***	n/a	-46 ***

Table 4: RMS VAO model forecast errors less RMS random walk forecast errors, by horizon and operating regime. A negative entry (shaded) indicates VAO model outperformance, and ***, **, *, ^^, and ^ represent 1, 5, 10, 20, and 40 percent two-tailed levels of significance using the Diebold and Mariano (1995) method.

Item	Forecast error	FF	GS1	GS3	GS5	GS10	GS30	FF/ GS10
No term premium	Mean	34	82	88	91	83	14	50
	RMS	123	149	144	136	119	74	109
	RMS r/t RW	-35 ^^	12 ^	21 *	24 **	25 **	0	-53 *
With term premium	Mean	-8	-32	-21	5	23	-31	30
	RMS	118	128	116	100	87	78	102
	RMS r/t RW	-40 ^	-9	-7	-11 ^	-7 *	5	-61 **

Table 5: 1994 to 2003 VAO model forecast error analysis for the one-year horizon. The last line in each block is the RMS VAO model forecast error less the RMS random walk forecast error; a negative entry (shaded) indicates VAO model outperformance with levels of significance as noted in table 4.
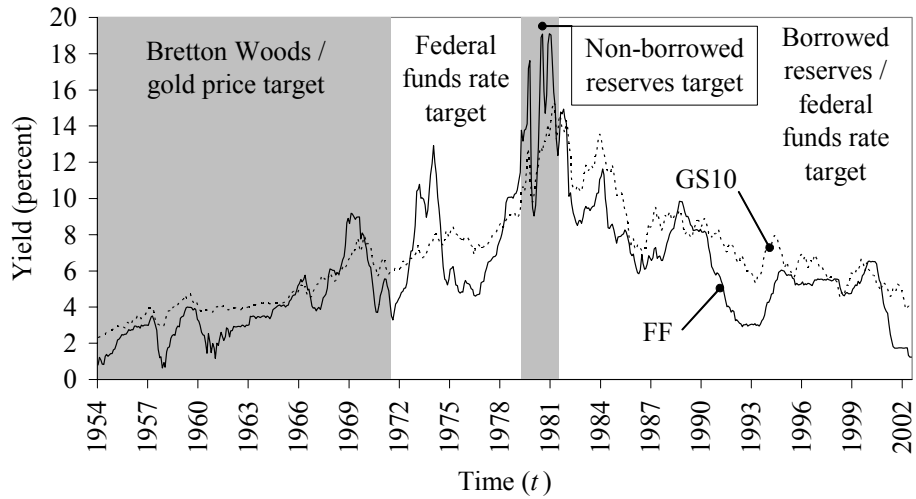
Figure 3: The time series of the federal funds rate (FF) and the 10-year government bond yield (GS10). The shading indicates the four different monetary policy regimes that prevailed over the sample, which are only labelled in this figure.
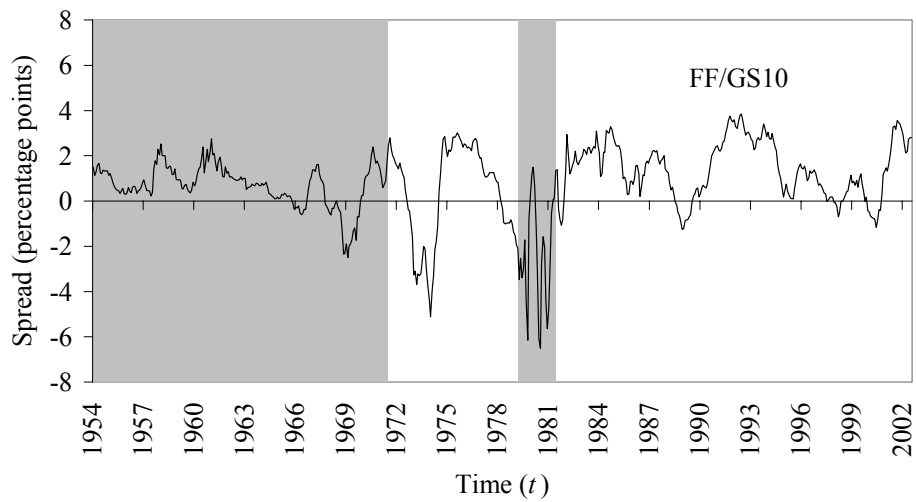

Figure 4: The time series of the 10-year government bond yield (GS10) less the federal funds rate (FF), denoted as FF/GS10.
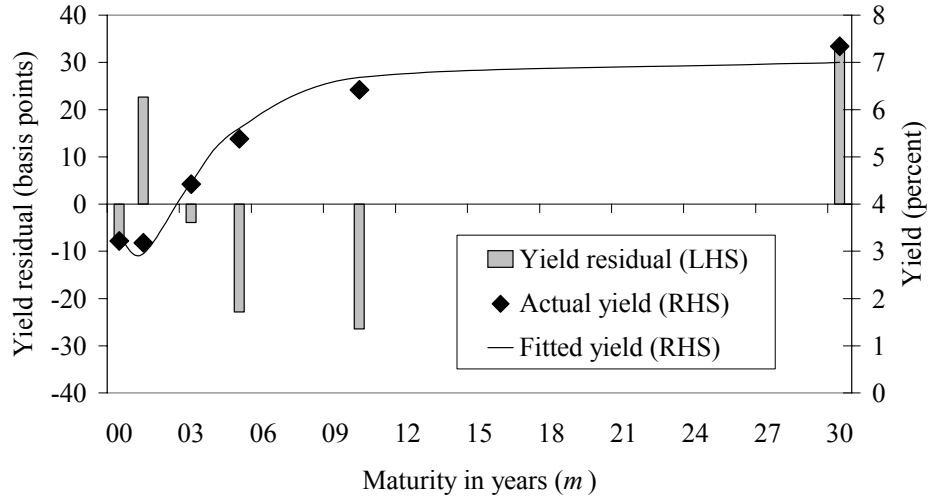
49

Figure 5: US yield curve data for $t =$ September 1992, and the cross-sectional fit of the VAO model. The estimated coefficient vector is $\boldsymbol{\beta}\,(\text{Sep-92}) = \{8.31, 9.19, -4.23\}'$ percentage points, and the volatility estimate for the full sample is $\boldsymbol{\sigma} = \{0.75, 2.23, 1.85\}'$ percentage points. The RMS yield residual for FF to GS10 is 18.9 basis points, which is the September 1992 data point in figure 6.
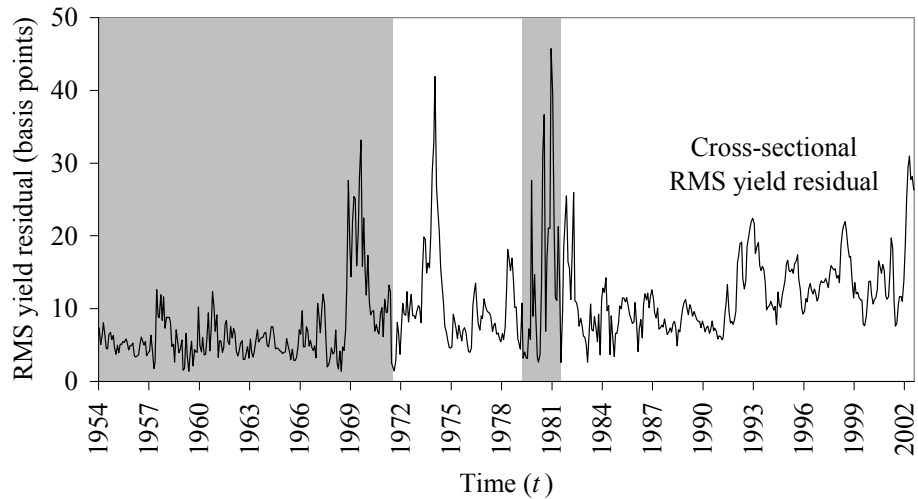
Figure 6: The time series of the RMS yield residuals for FF to GS10 for each cross-sectional fitting of yield curve data. The mean of this series is Cross-section 1 in table 1.
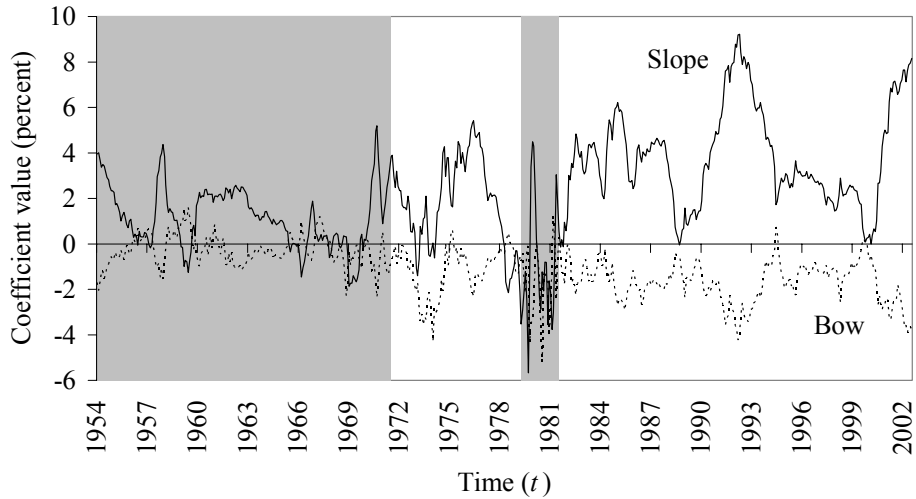
Figure 7: The time series of estimated Slope and Bow coefficients over the full sample. The volatility estimate for the full sample is $\boldsymbol{\sigma} = \{0.75, 2.23, 1.85\}'$ percentage points.
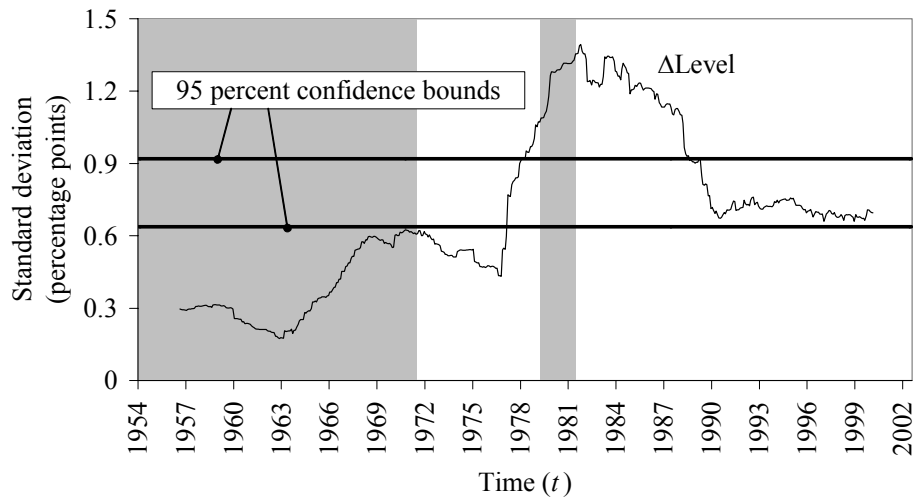

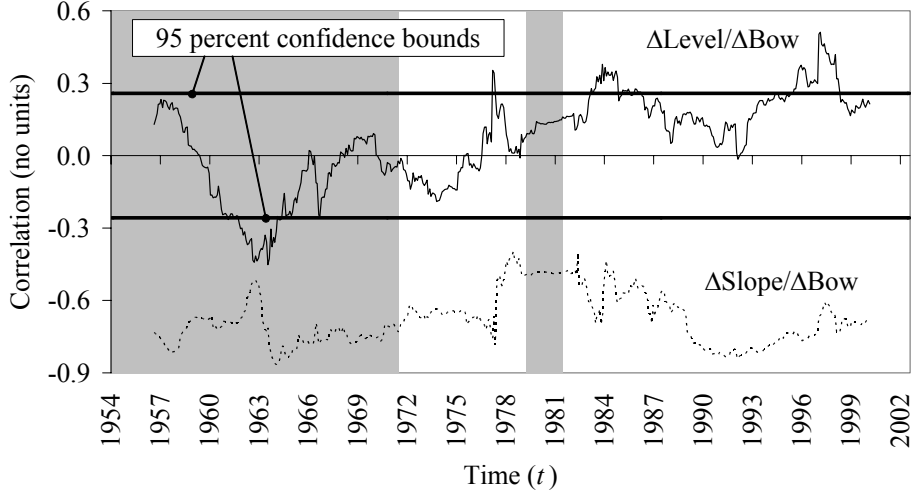Figure 8: Annualised standard deviations of monthly changes in the Level coefficient calculated over five-year centred moving-average windows. The 95 percent confidence interval assumes the null hypothesis that $\sigma_1 = 0.75$ percentage points (the annualised standard deviation of changes for the entire sample).

51

Figure 9: Correlations of changes in the Level, Slope and Bow coefficients calculated over five-year centred moving-average windows. The 95 percent confidence interval assumes the null hypothesis of zero correlation.
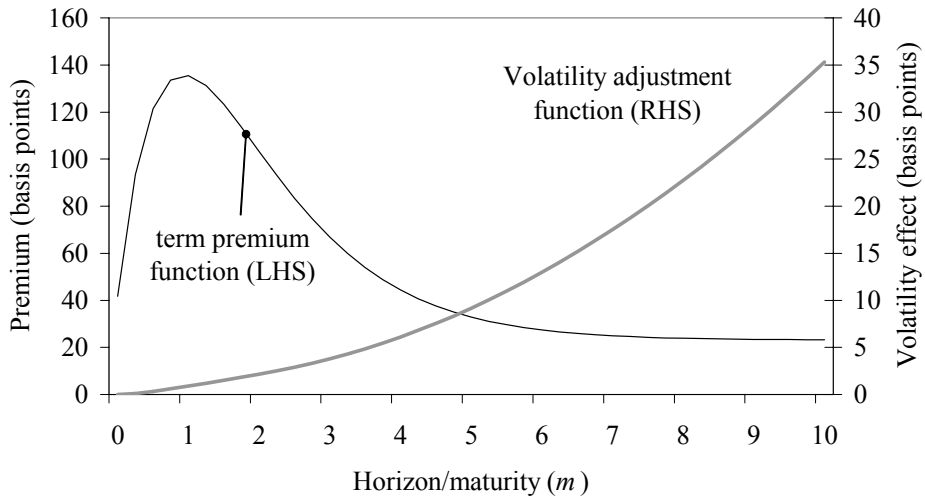

Figure 10: The estimated term premium function $\boldsymbol{\mu}'\mathbf{g}(\phi, m)$ and the volatility adjustment function $\mathbf{v}'\mathbf{h}(\phi, m)$ as functions of horizon/maturity. $\boldsymbol{\mu} = \{0.23, -1.62, 1.43\}'$ percentage points, and $\boldsymbol{\sigma} = \{0.84, 1.40, 1.12\}'$ percentage points.
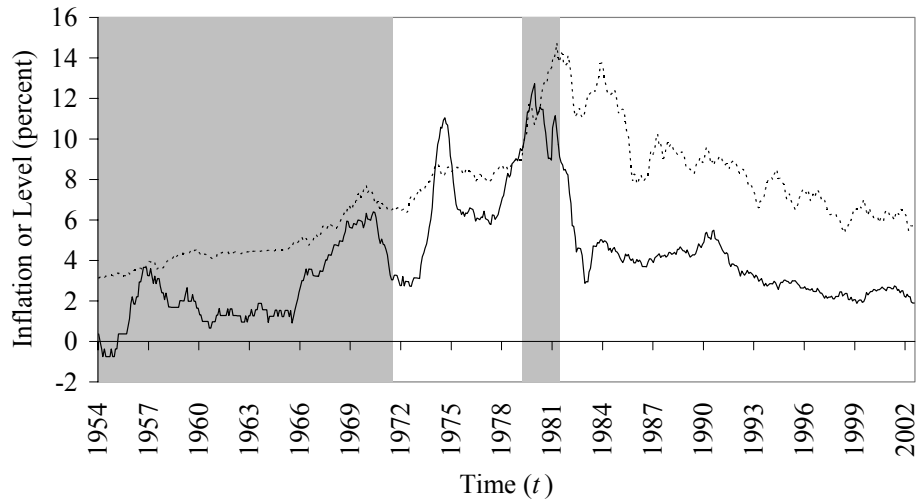
Figure 11: The time series of CPI inflation and the estimated Level coefficient over the full sample. The volatility estimate for the full sample is $\boldsymbol{\sigma} = \{0.75, 2.23, 1.85\}'$ percentage points.
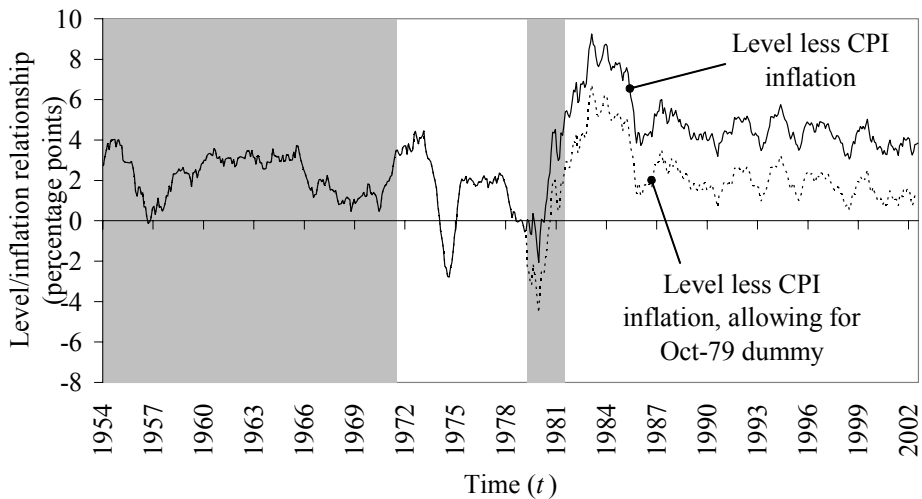

Figure 12: The time series of the Level coefficient less CPI inflation. The dotted line is the residuals from the Level coefficient less CPI inflation series regressed on a step dummy variable from October 1979.

Model	Estimated parameters	Unit root test	Cointegration tests	
		CPI inflation	Level less CPI inflation	Including Oct-79 dummy
Constant	AR1 coefficient	0.993	0.987	0.977
volatility	Phillips-Perron statistic	-2.32	-2.70 *	-3.62 **
VAO	Window bandwidth	18	18	18
5-year m/a	AR1 coefficient	0.993	0.988	0.977
volatility	Phillips-Perron statistic	-2.32	-2.56	-3.58 **
VAO	Window bandwidth	18	18	18

Table 6: Phillips-Perron unit root tests on the time series of CPI inflation plotted in figure 11, and the two Level coefficient less CPI inflation series plotted in figure 12.