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Estimating and Applying Autoregression Models Via Their Eigensystem Representation

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Abstract

This article introduces the eigensystem autoregression (EAR) framework, which allows an AR model to be specified, estimated, and applied directly in terms of its eigenvalues and eigenvectors. An EAR estimation can therefore impose various constraints on AR dynamics that would not be possible within standard linear estimation. Examples are restricting eigenvalue magnitudes to control the rate of mean reversion, additionally imposing that eigenvalues be real and positive to avoid pronounced oscillatory behavior, and eliminating the possibility of explosive episodes in a time-varying AR. The EAR framework also produces closed-form AR forecasts and associated variances, and forecasts and data may be decomposed into components associated with the AR eigenvalues to provide additional diagnostics for assessing the model.

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autoregression
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JEL Classification

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1 Introduction

In this article, I develop a framework for specifying, estimating, and applying an autoregression model (AR) explicitly using its associated eigenvalues and eigenvectors. What I will hereafter refer to as the eigensystem AR (EAR) framework therefore allows AR dynamics to be structured in ways that should prove useful for theoretical and empirical time series applications, such as in the collection of examples I introduce further below, but which would not be possible within ARs specified and estimated as a linear system.

As context to the above, an AR(P) linearly relates a variable to its own P lagged values via a coefficient vector $\phi = [\phi_1, \ldots, \phi_P]$, i.e. $y_t = \alpha + \phi_1 y_{t-1} + \ldots + \phi_P y_{t-P} + \varepsilon_t$. That specification allows conditional maximum likelihood estimates to be obtained via Ordinary Least Squares (OLS), e.g. see Hamilton (1994) chapter 5, and so I will hereafter refer to such a specification and its estimation as an OLS AR (OAR). The nature of the dynamics for an OAR is determined by its eigenvalues $\lambda = [\lambda_1, \ldots, \lambda_P]$, which may be obtained by factoring the polynomial associated with the AR or using the companion matrix formed with the estimated AR coefficient vector, e.g. see Hamilton (1994) Appendix 1.A. For example, if all AR eigenvalues have an absolute value less than 1, then the OAR is stationary and therefore has mean-reverting dynamics. If any eigenvalue has an absolute value greater than 1 then the OAR has explosive dynamics, which may be moderate in the sense of an AR(1) with $\phi_1 = 1 + c/k_T$ as in Phillips and Magdalinos (2007) or local to unity if $k_T = T$. The intermediate case is an OAR with a unit root, which corresponds to an eigenvalue with an absolute value of 1. In addition to AR dynamics being stationary, unit root, or explosive, the presence of complex or negative real eigenvalues will overlay oscillatory dynamics in some form.

The EAR framework essentially works as the reverse of the OAR description above. That is, the AR(P) is specified directly via P eigenvalues within lag polynomial eigenvalue factors, and their product is calculated using vector convolution to obtain the coefficients for the AR(P). The eigenvalues may therefore be estimated with any equality and/or inequality constraints. Such eigenvalue constraints cannot generally be achieved with linear restrictions on the AR coefficients within an OAR.[1]

Because the eigenvalues of an AR determine the nature of its dynamics, as discussed earlier, EAR estimation may therefore be structured to deliver allowable AR dynamics as may be required and/or desired for the task at hand. Examples illustrated in this article include EAR estimations with the following constraints: (1) all $|\lambda_k| \leq 1 + 1/T$, to ensure that any explosiveness is local to unity; (2) all $|\lambda_k| < 1$, to ensure non-explosive behavior; (3) only real positive eigenvalues to avoid pronounced oscillatory dynamics; (4) a unit root in complex/oscillatory form, to illustrate imposed non-decaying periodicity; and (5) repeated real eigenvalues, to illustrate an avenue of imposing parsimony. Any of the eigenvalue constraints above may be applied in a time-varying context, and I estimate an example of a time-varying EAR with the constraint $|\lambda_k| < 1$ that ensures non-explosive dynamics at all times.

The EAR framework also provides a useful basis for applying any AR, including an

[1] The AR(1) to be discussed in section 2.1 is a trival exception. Another well-known case is that a real eigenvalue equal to 1 may be imposed by estimating an OAR with first differenced data (which is a particular case of the EAR estimation method outlined in section 4.3.4). In general, as I show in section 3, the relationship between the coefficients and the eigenvalues for an AR is inherently non-linear and becomes more so as the order is increased.

OAR once its eigensystem is obtained. That is, closed-form expressions for the forecasts and impulse response functions (IRFs) of an AR, along with their confidence intervals, may be obtained for any horizon h, rather than using the typical recursive approach. The closed-form expressions also show that AR forecasts/IRFs may be decomposed into a sum of AR(1) and AR(2) components (plus additional components associated with repeated eigenvalues if that constraint is imposed in an EAR). The data used to estimate the AR may itself also be decomposed into historical AR(1) and AR(2) components. These historical and forecast/IRF decompositions and their associated variances provide diagnostic perspectives on the dynamics of an AR that are not apparent from the coefficients or the undecomposed forecasts/IFS.

Literature related to specifying, estimating, and applying an AR via its eigensystem appears to be limited. In the context of specifying time series models with given spectral properties, Boshnakov and Iqelan (2009) section 2.1 notes the well-known unique correspondence between an AR and the roots of its associated polynomial equation (i.e. the inverse of the eigenvalues), but does not develop a framework for model estimation on that basis. Likewise, the multivariate eigensystem specifications for a vector autoregression (VAR) outlined in Neumaier and Schneider (2001), Boshnakov (2002), and Krippner (2010) also do not include estimation frameworks based on the underlying eigensystem of the VAR. Neumaier and Schneider (2001) shows how a VAR may be re-expressed as a sum of AR(1) models, which is closely related to the historical decomposition developed in section 5 of the present article. However, the expressions are not extended to forecasts/IRFs or FEVs, and the complex components are not transformed into real AR(2) components.

The remainder of the article proceeds as follows. Section 2 outlines the aspects of the AR(1) and AR(2) that form the building blocks for subsequent sections of the article. Hence, section 3 presents a straightforward generalization of the AR(2) lag polynomial to specifying an AR(P) in terms of its eigenvalues and obtaining its coefficients. In section 4, I use the AR(1) and AR(2) building blocks to develop the conditional maximum likelihood estimation method for the EAR, and again in section 5 to derive the closed-form expressions for AR forecasts/IRFs, along with AR(1) and AR(2) component decompositions. Section 6 applies the EAR framework empirically, i.e. it illustrates estimations with the various eigenvalue constraints mentioned earlier, provides an example of decomposing historical data and forecasts into AR(1) and AR(2) components, and shows how a time-varying AR(P) may be estimated with non-explosive dynamics. Section 7 concludes with a brief summary, notes other potential applications that the EAR framework could be applied to, and then briefly discusses the extension to the multivariate context, i.e. considering vector autoregression models from an eigensystem perspective. The appendices contain most of the proofs for the propositions in the main text. For this working paper version, the appendices also contain additional material related to various sections of the paper, but which is not central to the developments in the main text.

2 AR(1) and AR(2) models

In this section, I provide expositions of the AR(1) and AR(2) that are directly relevant to subsequent sections of the article. Most aspects are elementary and available in standard texts, for example Hamilton (1994) chapters 1 and 2. However, I include for the AR(2) a proposition used later in the article, and also an incidental proposition that I have not seen elsewhere.

2.1 The AR(1) and its dynamics

The AR(1) is typically represented in OLS regression form as follows:

$$y_t = \alpha + \phi_1 y_{t-1} + \varepsilon_t \tag{1}$$

where y_t, and y_{t-1} are respectively the data at times t and $t-1$, α is the constant parameter, ϕ_1 is the AR(1) coefficient, and ε_t are the residuals, which are assumed to at least have the properties $\mathbb{E}[\varepsilon_t] = 0$, and $\mathbb{E}[\varepsilon_t \varepsilon_s] = \Omega_\varepsilon$ if $t = s$ and zero otherwise, or more typically ε_t is simply assumed to be iid normal, i.e. $\varepsilon_t \sim N(0, \Omega_\varepsilon)$.

An alternative representation of the AR(1) is the lag polynomial form:

$$(1 - \phi_1 L)(y_t - \mu) = \varepsilon_t \tag{2}$$

where L is the lag (or backshift) operator so that $Ly_t = y_{t-1}$ and $\mu = \alpha/(1 - \phi_1)$ is the process mean (if the model is stationary/mean-reverting, i.e. $|\phi_1| < 1$).

The eigenvalue for the AR(1) is $\lambda_1 = \phi_1$, which must be real. When λ_1 is positive, the IRF for the AR(1) will be a simple exponential function of the three types in figure 1, i.e. exponential growth or explosive if $\lambda_1 > 1$, an exponential decay or stationary/mean-reverting if $0 < \lambda_1 < 1$, and static or unit root in the intermediate case of $\lambda_1 = 1$. When λ_1 is negative, the magnitude of its component will be as for the positive cases, but it will oscillate in sign each period (i.e. with a wavelength of 2 periods).

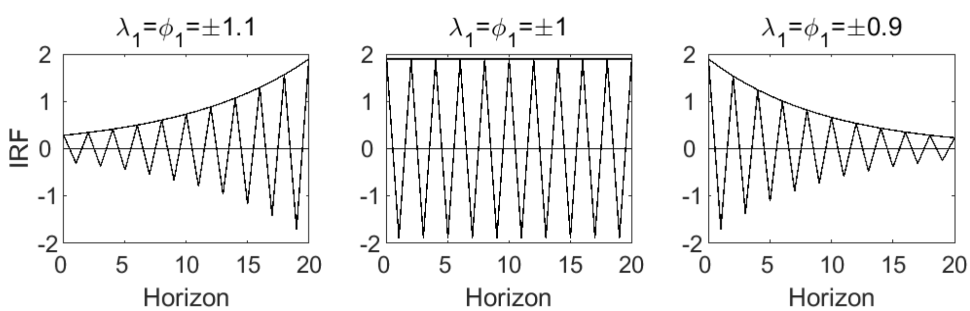

Figure 1: The three panels contain the six types of dynamics for AR(1) models (i.e. explosive, unit root, and stationary/mean-reverting, each with positive or negative coefficients) represented as IRFs standardized to a maximum of 1.9 units.

2.2 The AR(2) and its dynamics

The AR(2) is typically represented in OLS regression form as follows:

$$y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t \tag{3}$$

where y_t, y_{t-1}, and y_{t-2} are respectively the data at times t, $t-1$, and $t-2$, and ϕ_1 and ϕ_2 are the pair of AR(2) coefficients.

The representation of the AR(2) in its lag polynomial form is:

$$\left(1 - \phi_1 L - \phi_2 L^2\right)(y_t - \mu) = \varepsilon_t \tag{4}$$

where $Ly_t = y_{t-1}$ and $L^2 y_t = y_{t-2}$, and $\mu = \alpha / (1 - \phi_1 - \phi_2)$ is the process mean (if the model is stationary). From this point onward, I set α and μ to zero for clarity in the specifications, and this also aligns with my subsequent empirical application to mean-adjusted data in section 6. However, either α or μ could be included within any of the AR models in this article as an unconstrained parameter to be estimated in conjunction with the other AR parameters.

Equation 4 may be factored as for a quadratic polynomial, i.e.:

$$(1 - \lambda_1 L)(1 - \lambda_2 L) y_t = \varepsilon_t \tag{5}$$

where $(1 - \lambda_1 L)$ and $(1 - \lambda_2 L)$ are the eigenvalue factors, and:

$$(\lambda_1, \lambda_2) = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} \tag{6}$$

are the two eigenvalues for the AR(2). The eigenvalues will be real and distinct if $\phi_1^2 + 4\phi_2 > 0$, repeated if $\phi_1^2 + 4\phi_2 = 0$, or a complex conjugate pair if $\phi_1^2 + 4\phi_2 < 0$, i.e. $(\lambda_1, \lambda_2) = \left(\lambda_1, \overline{\lambda_1}\right) = \frac{1}{2}\phi_1 \pm \frac{1}{2} i \sqrt{\phi_1^2 + 4\phi_2}$. Respectively adding and multiplying the two eigenvalues gives the original pair of AR(2) coefficients in terms of the eigenvalue pair, i.e.:

$$(\phi_1, \phi_2) = (\lambda_1 + \lambda_2, -\lambda_1 \lambda_2) \tag{7}$$

Another representation for the AR(2) is its companion form, i.e.:

$$\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix} \tag{8}$$

which I use in section 5 as part of of the basis for decomposing AR(P) forecasts/IRFs into components, as with the IRF examples in figure 2 below. Related to the latter, sections A.1 to A.3 in appendix A contain details specific to the AR(2) companion form and its forecasts/IRFs, including the case of repeated eigenvalues.

The types of possible dynamics for an AR(2) may be summarized as in the top two panels of figure 2,which respectively represent the AR(2) in terms of its coefficient point (ϕ_1, ϕ_2) or its eigenvalue pair (λ_1, λ_2). An explosive, mean-reverting, or unit root AR(2) respectively has (ϕ_1, ϕ_2) outside, inside, or on the edge of the plotted "stability triangle", which respectively corresponds to at least one eigenvalue outside the unit circle, both eigenvalues inside the unit circle, or at least one eigenvalue on the unit circle.

A point (ϕ_1, ϕ_2) below the parabola $\phi_2 = -\frac{1}{4}\phi_1^2$ is a region associated with oscillatory dynamics. Specifically, the IRF will be the sum of a sine and a cosine function with wavelengths of $2\pi/\theta$ periods, where θ is the angle in radians of the complex eigenvalue λ_1 in polar form, i.e. $\theta = \cos^{-1}(\mathrm{Re}[\lambda_1 / |\lambda_1|])$ or $\theta = \cos^{-1}\left(\phi_1 / \left[2\sqrt{-\phi_2}\right]\right)$. The bottom-left panel contains an IRF example from an oscillatory AR(2) with $\theta = 0.2\pi$ and a wavelength of 10 periods, and the sine and cosine components of the IRF are also plotted.

A point (ϕ_1, ϕ_2) on or above the parabola corresponds to both eigenvalues being on the real line, which produces an IRF that is the sum of two AR(1) IRFs, as illustrated in figure 1, each with one of the eigenvalues as the AR(1) coefficient. The bottom-right panel of figure 2 contains an IRF example from an AR(2) with two positive eigenvalues, and hence two exponential decay components that produce a "hump-shaped" profile. Note that a negative real eigenvalue is the limit of oscillatory dynamics with a wavelength of two periods, which is apparent from a negative number in complex polar form having the angle $\theta = \pi$, and hence a wavelength of 2 periods.

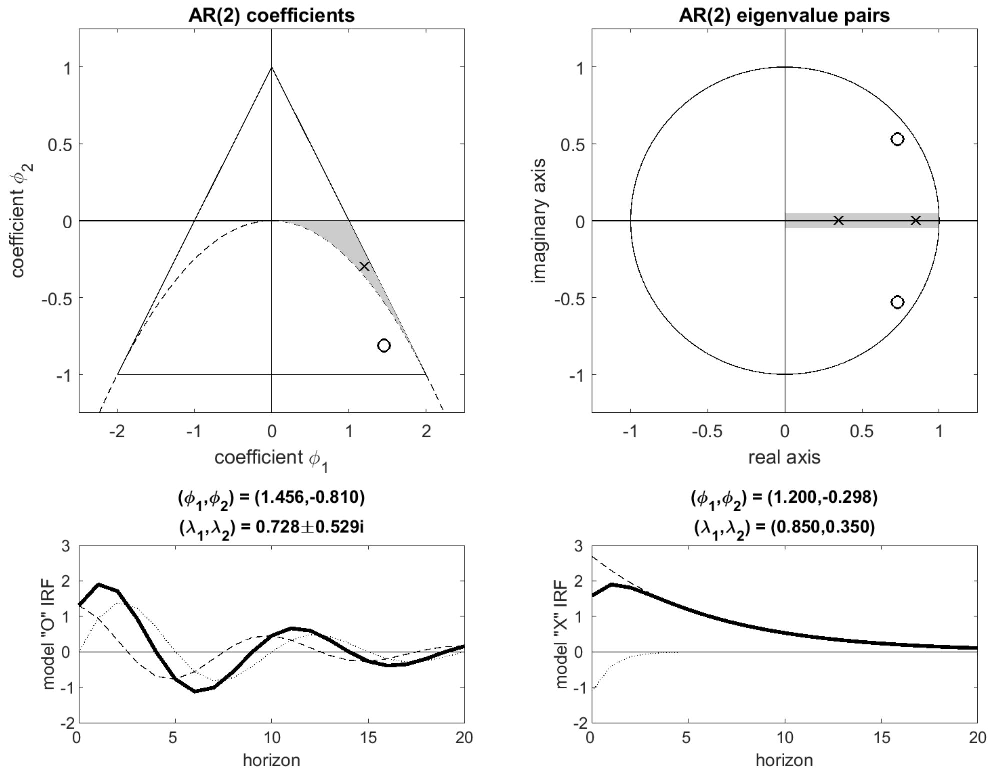

Figure 2: Two examples of AR(2) models. The top-left panel plots the AR(2) coefficients for the models as the points (ϕ_1, ϕ_2), along with the AR(2) stability triangle. The top-right panel plots the pairs of eigenvalues associated with the models, along with the unit circle. The two bottom panels respectively plot the IRFs from each model, with both standardized to a maximum of 1.9 units, along with their IRF components.

The stability triangle may be generalized, as in Proposition 1, to restrict the eigenvalues to be less than an arbitrary magnitude of γ. This generalization accommodates non-unity constraints on the magnitudes of AR(2) eigenvalues, and therefore on AR(P) models in general, as will be discussed in section 4. Proposition 2 notes that the region of the stability triangle (or its generalized version) associated with two real eigenvalues may be further divided into regions with two negative eigenvalues, one positive and one negative eigenvalue, or two positive eigenvalues.

Proposition 1 *The eigenvalues of an AR(2) may be constrained to an arbitrary magnitude γ using the following constraints on ϕ_1 and ϕ_2:*

$$\begin{aligned} |\phi_1| &< 2\gamma \\ \phi_2 &< \gamma(\gamma - |\phi_1|) \\ \phi_2 &> -\gamma^2 \end{aligned} \tag{9}$$

Proof. See section A.4 of appendix A. ■

Proposition 2 *The region of the AR(2) stability triangle in figure 1 with real eigenvalues, or the analogous region in the generalized triangle of proposition 1, may be further divided* *into regions with two positive eigenvalues (the shaded region in figure 1), two negative eigenvalues (the mirror image with respect to the ϕ_2 axis of the shaded region in figure 1), and one positive and one negative eigenvalue (the region above $\phi_2 = 0$ in figure 1).*

Proof. See section A.5 of appendix A. ■

3 Eigenvalue specification of an AR(P)

This section outlines how the eigenvalue representation underlying the AR(2) may be generalized to an AR with P lags, i.e. an AR(P). Section 3.1 begins with a standard representation of an AR(P) as the product of P eigenvalue factors, which results in a P-order lag polynomial. Section 3.2 outlines the mechanics of using vector convolution to generate the lag polynomial coefficients, and hence the AR(P) coefficients, from the eigenvalue factors.

3.1 AR(P) from P eigenvalues

Generalizing the AR(2) representation in equation 5, an AR(P) may be represented by P eigenvalues as follows:

$$\left[\prod_{k=1}^{P}(1-\lambda_k L)\right] y_t = \varepsilon_t \tag{10}$$

where $k = 1, \ldots, P$ are the index numbers for the eigenvalues $\lambda = [\lambda_1, \ldots \lambda_P]$, and $(1 - \lambda_k L)$ are lag polynomial eigenvalue factors, which I will hereafter refer to simply as eigenvalue factors. Analogous to equation 4, expanding the factored lag polynomial in equation 10 will produce a lag polynomial of order P, i.e.:

$$\left[1 - \sum_{p=1}^{P}\phi_p L^p\right] y_t = \varepsilon_t \tag{11}$$

where ϕ_p is the coefficient associated with the lag exponent L^p.[2] Applying the operators L^p to y_t thereby produces an OLS regression form analogous to equation 3. The following summarizes the various ways that the OLS regression form for the AR(P) will be expressed in this article, i.e.:

$$\begin{aligned} y_t &= \phi_1 y_{t-1} + \ldots + \phi_P y_{t-P} + \varepsilon_t \\ &= \left[\sum_{p=1}^{P}\phi_p y_{t-p}\right] + \varepsilon_t \\ &= [\phi_1, \ldots, \phi_P]\begin{bmatrix} y_{t-1} \\ \vdots \\ y_{t-P}\end{bmatrix} + \varepsilon_t \\ &= \phi Y_{t-1} + \varepsilon_t \end{aligned} \tag{12}$$

where the last form uses the compact notation $\phi = [\phi_1, \ldots, \phi_P]$ and $Y_{t-1} = [y_{t-1}, \ldots, y_{t-P}]'$.

[2] Hamilton (1994) pp. 33-34 contains equivalent expressions for equations 10 and 11, but in the reverse context, i.e. finding the eigenvalues for the lag polynomial (or its companion form).

3.2 AR(P) coefficients from vector convolution

Expanding the product of eigenvalue factors in equation 10 may be done via the vector convolution algorithm, which is routine for multiplying any two algebraic polynomials. That is, represent the respective polynomial coefficients as the m- and n-element vectors u and v, and then obtain the coefficients for the resulting polynomial as an $(m+n-1)$-vector w, where each element k of w is the result of the following summation:[3]

$$w(k) = \sum_{j=\max(1,k+1-n)}^{\min(k,m)} u(j)\, v(k-j+1) \tag{13}$$

The AR(2) from section 2.2 provides a convenient illustration. First, directly expanding the factored form of the lag polynomial in equation 5, i.e. $(1-\lambda_1 L)(1-\lambda_2 L)$, gives the result $1-(\lambda_1+\lambda_2)L+\lambda_1\lambda_2 L^2$. For the convolution, $(1-\lambda_1 L)$ and $(1-\lambda_2 L)$ are respectively represented as the two-element vectors $u=[1,-\lambda_1]$ and $v=[1,-\lambda_2]$, so $m=n=2$. The convolution of u and v then produces the three-element vector w_2 representing the coefficients for the AR(2) lag polynomial, i.e.:[4]

$$\begin{aligned} w_2 &= \text{conv}\left([1,-\lambda_1],[1,-\lambda_2]\right) \\ &= [1,-(\lambda_1+\lambda_2),\lambda_1\lambda_2] \end{aligned} \tag{14}$$

which, as already obtained from the direct expansion, are the coefficients associated with L^0 (i.e. 1), L^1, and L^2. The two AR(2) coefficients from the convolution are therefore $\phi_1=\lambda_1+\lambda_2$, and $\phi_2=-\lambda_1\lambda_2$, as in equation 7 from section 2.2.

The results for a higher-order AR, like for the EAR estimations outlined in section 4.2, may be obtained by continuing to iterate the convolution result with the next two-element vector, e.g. $w_3=\text{conv}(w_2,[1,-\lambda_3])$, and so on over all $[1,-\lambda_k]$ vectors. The resulting vector w_P will contain $1+P$ elements, with the first element being 1. Negating the remaining P elements obtains the coefficients $\phi=[\phi_1,\ldots,\phi_P]$ for equation 12.

Anticipating the discussion in section 4.3.4, the coefficients for an AR(P) may also be obtained by multiplying two lag polynomials of order $1+K$ and $1+P-K$, respectively. Representing the two lag polynomials as the vectors $u=[1,-\delta_1,\ldots,-\delta_K]$ and $v=[1,-\theta_1,\ldots,-\theta_{P-K}]$, the convolution $w_P=\text{conv}(u,v)$ will again be a vector with $1+P$ elements that contains the AR(P) coefficients $\phi=[\phi_1,\ldots,\phi_P]$.

4 OAR and EAR estimation

In this section, I outline how the parameters for a specified AR may be estimated from a time series of data. Section 4.1 introduces the conditional log-likelihood function that applies for the OAR and EAR, and section 4.2 discusses the conditional maximum likelihood estimation (CMLE) of the parameters for the OAR. In section 4.3, I present methods for CMLE of the EAR, and section 4.4 outlines how eigenvalue constraints may be incorporated into a time-varying EAR.

[3] The convolution function is "conv(u,v)" in MatLab, or otherwise the algorithm is straightforward to code as a double summation.

[4] For the AR(2) example in the text, the three summations to obtain the elements for w_2 are: $w_2(1)=\sum_{j=1}^{1}u(j)\,v(2-j)=1\cdot 1=1$, $w_2(2)=\sum_{j=1}^{2}u(j)\,v(3-j)=1\cdot -\lambda_2+-\lambda_1\cdot 1=-(\lambda_1+\lambda_2)$, and $w_2(3)=\sum_{j=2}^{2}u(j)\,v(4-j)=-\lambda_1\cdot -\lambda_2=\lambda_1\lambda_2$.

4.1 The conditional log-likelihood function

The log-likelihood function, conditioned on the initial P observations of the $1 \times (P+T)$ dataset $y = \{y\}_{1-P}^{T}$, for the EAR in equation 10 and the OAR in equation 12 is:[5]

$$\log\left(\mathcal{L}\left[\Theta, \Omega_\varepsilon\right]\right) = -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\Omega_\varepsilon) - \frac{1}{2\Omega_\varepsilon}\sum_{t=1}^{T}\varepsilon_t^2 \tag{15}$$

where the relevant parameter sets Θ that define $\varepsilon_t = \varepsilon_t(\Theta)$ and hence $\log\left(\mathcal{L}\left[\Theta, \Omega_\varepsilon\right]\right)$ are outlined in sections 4.2 and 4.3. The conditional log-likelihood function implicitly assumes that ε_t is iid normal, otherwise the subsequent estimation is quasi-CMLE. Also note that I undertake the CMLE on mean-adjusted data; if a constant α or mean μ were included in the specification, either would be added as an unconstrained parameter to be estimated at the same time as the parameter set $[\Theta, \Omega_\varepsilon]$.[6]

4.2 OAR estimation

For the OAR, ε_t in equation 12 is a linear function of the AR coefficients $\phi = \left[\phi_1, \ldots \phi_p\right]$, and $\varepsilon_t(\phi)$ is defined from $y_t = \phi Y_{t-1} + \varepsilon_t$ as in equation 12. Therefore, the CMLE of ϕ may be obtained by analytically maximizing the log-likelihood function, which obtains the following results:

$$\phi = yY'_{-1}\left(Y_{-1}Y'_{-1}\right)^{-1} \tag{16a}$$

$$\Omega_\varepsilon = \frac{1}{T}\varepsilon\varepsilon' \tag{16b}$$

where y is a $1 \times T$ vector containing all y_t (from 1 to T), Y_{-1} is a $P \times T$ matrix containing all Y_{t-1}, and ε is a $1 \times T$ vector containing all ε_t. Equation 16a is the typical coefficient vector of an AR obtained by an OLS regression, and equation 16b is the CMLE estimate of the variance (i.e. without the degrees of freedom adjustment for OLS estimation); e.g. see Hamilton (1994) pp. 125-26 or 295-96, and I have also included an exposition in section B.1 of appendix B.

4.3 EAR estimation

Regarding the EAR, even before any constraints on the eigenvalues are included, ε_t is a non-linear function of the eigenvalues $\lambda = [\lambda_1, \ldots \lambda_P]$. More specifically, originating from the product in equation 10, the coefficient vector $\phi = [\phi_1, \ldots, \phi_P]$ is a non-linear function of the eigenvalues $\lambda = [\lambda_1, \ldots \lambda_P]$, so:

$$y_t = \phi(\lambda) Y_{t-1} + \varepsilon_t \tag{17}$$

The CMLE of λ is therefore not attainable by analytically maximizing the log-likelihood function, and some method of numerical optimization is required. Hence, the computational expense of estimating an AR via the EAR framework is only worthwhile when eigenvalue constraints are desired and/or required.

[5] See, for example, Hamilton (1994) pp. 125-26.

[6] See Lütkepohl (2006) sections 3.3 and 3.4 for discussion on estimating the parameters of time series models with mean-adjusted data.

The following subsections outline three examples of estimating an EAR with eigenvalue inequality and equality constraints that I subsequently use in the empirical illustrations contained in section 6. The first example is the most straightforward case where the EAR eigenvalues are constrained to be Positive and Real with a maximum magnitude of γ, and so I hereafter refer to this example as the PREAR. The second example allows for Complex and Real (positive or negative) eigenvalues while respecting the magnitude constraint of γ, and so I refer to this more general case as the CREAR. Section 4.3.3 provides two examples of equality constraints. In sub-section 4.3.4, I present an estimation approach that combines the approaches from sections 4.3.1 to 4.3.3 with a supplementary OAR, and I refer to this as the Hybrid-CREAR, or H-CREAR. Section 4.3.5 discusses the computational efficiency of various EAR methods.

4.3.1 PREAR estimation

The constrained optimization for the PREAR CMLE may be represented as:

$$\arg\max_{\lambda,\Omega_\varepsilon} \log\left(\mathcal{L}\left[\lambda,\Omega_\varepsilon\right]\right) \text{ subject to all } 0<\lambda_k<\gamma \tag{18}$$

Using a scaled logistic function of unconstrained parameters x_k to produce $0<\lambda_k<\gamma$, i.e.:

$$\lambda_k=\frac{\gamma}{1+\exp\left(-x_k\right)} \tag{19}$$

allows the constrained PREAR optimization to be converted into an unconstrained optimization, i.e.:

$$\arg\max_{x,\Omega_\varepsilon} \log\left(\mathcal{L}\left[x,\Omega_\varepsilon\right]\right) \tag{20}$$

where $x=[x_1\ldots,x_P]$, and the residuals ε_t in the log-likelihood function are now a non-linear function of x, i.e.:

$$\varepsilon_t=y_t-\phi\left(\lambda\left[x\right]\right)Y_{t-1} \tag{21}$$

where $\phi\left(\lambda\left[x\right]\right)$ explicitly denotes the dependence from the parameters x to the eigenvalues λ, and then to the AR coefficients, i.e. $\phi\left(\lambda\left[x\right]\right)=[\phi_1,\ldots,\phi_P]$.

While any numerical method could be used for the optimization to obtain x and Ω_ε, the squared residuals in the log-likelihood function make non-linear least squares (NLS) highly applicable. Section B.2 in appendix B provides further details. Furthermore, section B.3 shows that the simple functional forms used to define the PREAR readily allows a concise analytic calculation of the Jacobian matrix $\partial\left[\phi\left(x\right)\right]'/\partial x'$ for the NLS estimation. Otherwise a suitable Jacobian may be obtained using numerical derivatives, as available in typical optimization functions (e.g. within the MatLab function "lsqnonlin" that I use).

Regarding the starting values for x, I arbitrarily chose linear spacing between -2.197 and 2.944 for $[x_1,\ldots,x_P]$ in the examples of section 6, where -2.197 and 2.944 are the values that would produce 0.1 and 0.95 from equation 19. I use an arbitrary function tolerance of 1e-10 as the convergence criterion.

4.3.2 CREAR estimation

The constrained optimization for the CREAR CMLE may be represented as:

$$\arg\max_{\lambda,\Omega_\varepsilon} \log\left(\mathcal{L}\left[\lambda,\Omega_\varepsilon\right]\right) \text{ subject to all } \left|\lambda_k\right|<\gamma \tag{22}$$

Converting the constrained CREAR optimization to an unconstrained estimation is more involved than for the PREAR, because the transformation needs to allow for each λ_k to be either real or within a complex pair, but without knowing in advance which of those two cases applies. The generalized AR(2) triangle derived in section 2 offers a convenient geometric method for delivering pairs of AR(2) coefficients $\left(\phi_k^*, \phi_{k+1}^*\right)$,[7] which in turn produce eigenvalues that can be positive, negative, or complex conjugate pairs, while also ensuring that $|\lambda_k| < \gamma$. Hence, I use the following shifted scaled logistic function of the unconstrained parameter x_k:

$$\phi_k^* = 2\gamma \left(\frac{2}{1 + \exp\left(-x_k\right)} - 1 \right) \tag{23}$$

to first obtain a value for ϕ_k^* such that $-2\gamma < \phi_k < 2\gamma$ (which is the range of allowable values for the base of the generalized AR(2) triangle). Given ϕ_k^*, to ensure that $\left(\phi_k^*, \phi_{k+1}^*\right)$ falls within the generalized AR(2) triangle, the upper constraint for ϕ_{k+1} (which I denote $\overline{\phi}_{k+1}^*$) needs to be:

$$\overline{\phi}_{k+1}^* = \gamma \left(\gamma - |\phi_k^*|\right) \tag{24}$$

and the lower constraint for ϕ_{k+1}^* is $-\gamma^2$. Therefore, given an unconstrained parameter x_{k+1}, the following scaled shifted logistic function:

$$\phi_{k+1}^* = \frac{\overline{\phi}_{k+1}^* + \gamma^2}{1 + \exp\left(-x_{k+1}\right)} - \gamma^2 \tag{25}$$

will ensure a value of ϕ_{k+1}^* so that $\left(\phi_k^*, \phi_{k+1}^*\right)$ falls within the generalized AR(2) triangle. Equation 6 is then used to obtain $(\lambda_k, \lambda_{k+1})$ from $\left(\phi_k^*, \phi_{k+1}^*\right)$.[8]

Note that if there is an odd number of lags (and hence eigenvalues) for the CREAR, then the single remaining unpaired eigenvalue λ_P must be real, but it could be either negative or positive, i.e. $-\gamma < \lambda_P < \gamma$. Such a result is readily obtained from an unconstrained value x_P within the following shifted scaled logistic function:

$$\lambda_P = \gamma \left(\frac{2}{1 + \exp\left(-x_P\right)} - 1 \right) \tag{26}$$

Based on the discussion above, the constrained EAR optimization may be converted into an unconstrained optimization analogous to the PREAR, i.e.:

$$\arg \max_{x, \Omega_\varepsilon} \log \left(\mathcal{L}\left[x, \Omega_\varepsilon\right]\right) \tag{27}$$

except the parameters $x = [x_1 \ldots, x_P]$ are transformed differently to obtain the residuals $\varepsilon_t = y_t - \phi\left(x\right) Y_{t-1}$. Like for the PREAR, NLS estimation is again highly applicable, so

[7] The notation $\left(\phi_k^*, \phi_{k+1}^*\right)$ avoids any confusion with the CREAR coefficients themselves, i.e. $\left(\phi_k, \phi_{k+1}\right)$ from $\phi\left(\lambda\left[x\right]\right) = \left[\phi_1, \ldots, \phi_P\right]$. The latter are ultimately calculated from the full set of eigenvalue pairs $(\lambda_k, \lambda_{k+1})$, with each pair having been obtained using the generalized AR(2) triangle method. Note also that Morley (1999) provides an algebraic method for constraining AR(2) to be within the unit circle, which could also be generalized to allow for maximum eigenvalue magnitudes of γ.

[8] The $\left(\phi_k^*, \phi_{k+1}^*\right)$ pairs could also be used directly in the ultimate convolution to obtain $\left[\phi_1, \ldots, \phi_P\right]$. This would use convolutions of the vector $\left[1, -\phi_k^*, -\phi_{k+1}^*\right]$, as in section B.4 of appendix B, rather than the pairs of vectors $[1, -\lambda_k]$ and $[1, -\lambda_{k+1}]$.

the details in section B.2 of appendix B also apply to the CREAR. Similarly, section B.4 in appendix B shows that the CREAR functional forms are still simple enough to produce a relatively concise analytic calculation of the Jacobian matrix $\partial [\phi(x)]' / \partial x'$.

Arbitrary starting values can be used for x, but if a CREAR is being used to constrain an existing OAR (e.g. to constrain an OAR eigenvalue that is found to be explosive), then the eigenvalues from OAR estimates that are already available may be used to obtain better starting values. The steps are as follows: (1) any OAR eigenvalues with $|\lambda_k| > \gamma$ should be set to have $|\lambda_k| < \gamma$ (I use 0.99γ of the original λ_k); (2) convert the $(\lambda_k, \lambda_{k+1})$ pairs to (ϕ_k^*, ϕ_{k+1}^*) pairs using equation 7; and convert the (ϕ_k^*, ϕ_{k+1}^*) pairs to unconstrained (x_k, x_{k+1}) pairs using equations 23 to 25.

4.3.3 EAR estimation with equality constraints

The first example of EAR estimation with an equality constraint is setting the magnitude of a complex conjugate pair of eigenvalues to unity, which imposes an oscillatory unit root on the AR(P), as in examples 6 and 16 of section 6. With reference to the stability triangle in figure 2, this constraint may be imposed by setting $\phi_2^* = -1$ and using a shifted scaled logistic function to obtain $-2 < \phi_1^* < 2$ or, equivalently, using the CREAR method with $\gamma = 1$ and $x_2 = -\infty$.[9]

The second example of an equality constraint is a pair of repeated eigenvalues, e.g. $\lambda_1 = \lambda_2$ (which are necessarily real) as in examples 7, 10, 17, and 20 of section 6. A pair of repeated eigenvalues is readily incorporated into any EAR as a multiple of identical factors, e.g. $(1 - \lambda_1 L)(1 - \lambda_1 L)$ in the case of $\lambda_1 = \lambda_2$, and a magnitude constraint of γ may be also be imposed, i.e. using equation 19 with an unconstrained parameter x_k. In general, an EAR could be specified to repeat one eigenvalue more than twice and/or include more than one group of repeated eigenvalues. However, the presence of any repeated eigenvalues has implications for the forecast/IRF and historical decompositions outlined in section 5, as I will discuss in section 5.6.

4.3.4 H-CREAR estimation

In practice, only a subset of one or several eigenvalues in a CREAR estimation may need to be set to an equality constraint/s or be bound by an inequality constraint. In the former case, as formalized in Proposition 3 below, K eigenvalue equality constraints may be incorporated into the data itself, which is then used to estimate a supplementary OAR that implicitly contains the remaining unconstrained eigenvalues. Those eigenvalues could be explicitly calculated from the supplementary OAR coefficients if desired, but the AR(P) can be obtained directly from the constrained eigenvalues and the supplementary OAR coefficients. Note that, because the supplementary OAR provides no eigenvalue constraints, the H-CREAR result will generally contain both complex and real eigenvalues.

Proposition 3 *The coefficients of an AR(P) subject to K eigenvalue equality constraints*

[9] A point on the edge of the stability triangle imposes a unit root, and the bottom edge is in the region that produces complex conjugate eigenvalues and therefore oscillatory AR(2) dynamics. An alternative would be to use the the polar form for the first two eigenvalues, i.e. $(\lambda_1, \lambda_2) = r\exp(\pm\theta i)$, impose $r = 1$, and allow θ to be estimated within the range$-\pi < \theta < \pi$ using an unconstrained parameter within a scaled logistic function.

may be estimated using a lag polynomial generated from K eigenvalues, i.e.:

$$\prod_{k=1}^{K}(1-\lambda_k L)=\left[1-\sum_{p=1}^{K}\delta_p L^p\right] \tag{28}$$

and a supplementary OAR with $P-K$ lags, i.e.:

$$z_t=\sum_{p=1}^{P-K}\theta_p z_{t-p}+\varepsilon_t \tag{29}$$

where z_t is a variable obtained from the data y_t and the generated coefficients $\delta=[\delta_1,\ldots,\delta_K]$ as follows:

$$z_t=y_t-\sum_{p=1}^{K}\delta_p y_{t-p} \tag{30}$$

The AR coefficients $\phi=[\phi_1,\ldots,\phi_P]$ may then be obtained from the convolution $w=conv(u,v)$ of the two vectors $u=[1,-\theta_1,\ldots,-\theta_{P-K}]$ and $v=[1,-\delta_1,\ldots,-\delta_K]$.

Proof. See section B.5 of appendix B. ■

A familiar example of an eigenvalue equality constraint, as mentioned in footnote 1 from the introduction, is imposing a single real unit root on an AR by estimating an OAR applied to the time series of the first difference of y_t. Specifically, setting $\lambda_1=1$ and $K=1$ within the H-CREAR gives $\prod_{k=1}^{1}(1-\lambda_k L)=(1-L)$, and so $\delta_1=1$ and $z_t=y_t-y_{t-1}=\Delta y_t$. The coefficients $\theta=[\theta_1,\ldots,\theta_{P-1}]$ would be those obtained by an OAR applied to the time series of the first difference of y_t, i.e. $z_t=\sum_{p=1}^{P-K}\theta_p z_{t-p}+\varepsilon_t$ becomes $\Delta y_t=\theta_1\Delta y_{t-1}+\ldots+\theta_{P-1}\Delta y_{t-P+1}+\varepsilon_t$. A more general equality constraint is obtained by specifying the product $\prod_{k=1}^{K}(1-\lambda_k L)$, using convolution to obtain the coefficients $\delta=[\delta_1,\ldots,\delta_K]$, and then calculating $z_t=y_t-\delta_1 y_{t-1}-\ldots-\delta_K y_{t-K}$ to use in the supplementary OAR.

The case of an inequality constraint proceeds analogous to the equality case, except the estimation algorithm at each iteration provides the subset of eigenvalues $[\lambda_1,\ldots,\lambda_K]$. For example, if the largest eigenvalue from an OAR is real and positive with a value $\lambda_1>\gamma$, and all other eigenvalues have magnitudes $|\lambda_k|\ll\gamma$, then the constraint of $|\lambda_k|<\gamma$ will only bind for λ_1. The algorithm for the H-CREAR estimation therefore estimates just the eigenvalue λ_1, or more precisely x_1 that determines λ_1 subject to the constraint γ, with the supplementary OAR estimated by OLS within the log-likelihood function. A more general inequality constraint would estimate $[x_1,\ldots,x_K]$ that determines $[\lambda_1,\ldots,\lambda_K]$.

4.3.5 Computational efficiency

The closed-form analytic optimization underlying an OAR estimation will always be computationally less expensive than the numerical optimization underlying an EAR estimation. The latter will therefore be redundant unless there are potential benefits from imposing eigenvalue constraints for a given application. If only equality constraints are desired/required for the AR(P), then the H-CREAR allows the eigenvalues may be imposed while predominantly retaining the computationally efficient OAR estimation (which implicitly estimates the remainder of the eigenvalues without constraints). The H-CREAR also allows inequality constraints to be imposed on just one or several eigenvalues while estimating the remainder with an OAR, which is more efficient than imposing the inequality constraints on all eigenvalues. The PREAR necessarily requires an eigenvalue inequality constraint to apply on all eigenvalues, because the supplementary OAR estimation in the H-CREAR will in general produce complex and real eigenvalues.

4.4 Time-varying EAR

The OAR and EAR specifications and estimation methods outlined in section 4 implicitly assume that the AR(P) coefficients and eigenvalues are time-invariant parameters. But any of those specifications may be generalized to allow the coefficients and eigenvalues to vary over time, which I will denote as the time-varying (TV) OAR or EAR, i.e. the TVOAR or TVEAR. Allowing the parameters of an AR(P) to vary over time may better represent changing relationships/properties in the data, analogous to time-varying vector autoregressions in the multivariate case (e.g. Primiceri 2005).

The advantage of the TVEAR over the TVOAR is that the former allows direct constraints on the eigenvalues, which can be used to ensure that the time-varying AR(P) will always remain consistent with the allowable dynamics that may be required/desired. To illustrate this point, I first specify a simple TVOAR and a closely related TVEAR, and then discuss their differences in principle. I apply the TVOAR and TVEAR empirically in section 6.3.

The TVOAR in state space form is:

$$\phi_t = \phi_{t-1} + \eta_t \; ; \; \eta_t \sim N(0, \kappa I_P) \tag{31a}$$

$$y_t = \phi_t Y_{t-1} + \varepsilon_t \; ; \; \varepsilon_t \sim N(0, \Omega_\varepsilon) \tag{31b}$$

where y_t, Y_{t-1}, ε_t, and Ω_ε are as previously defined in section 4.2, and the coefficient vector ϕ_t is now a $1 \times P$ time-varying vector of state variables. This specification assumes that the coefficients in ϕ_t follow independent random walks, i.e. the state equation transition matrix is implicitly the $P \times P$ identity matrix I_P, and η_t is a normally distributed $1 \times P$ vector of innovations where κI_P sets the variances to κ and the covariances to be zero. The state equation is therefore defined with the single parameter κ, which governs the diffusion rate of the random walks for each of the state variables.

The specification for the TVEAR is:

$$x_t = x_{t-1} + \eta_t \; ; \; \eta_t \sim N(0, \kappa I_P) \tag{32a}$$

$$y_t = \phi(\lambda[x_t, \gamma]) Y_{t-1} + \varepsilon_t \; ; \; \varepsilon_t \sim N(0, \Omega_\varepsilon) \tag{32b}$$

where x_t is a $1 \times P$ vector of state variables which, in conjunction with the pre-defined parameter γ, defines the $1 \times P$ time-varying vector of coefficients $\phi(\lambda[x_t, \gamma])$. Specifically, analogous to time-invariant specifications in section 4.3, the P state variables from $x_t = [x_{1,t}, \ldots, x_{P,t}]$ define the set of P eigenvalues $\lambda_t = [\lambda_{1,t}, \ldots, \lambda_{P,t}]$, via the CREAR method, which in turn define the set of P coefficients $\phi(\lambda[x_t]) = [\phi_{1,t}, \ldots, \phi_{P,t}]$.

A setting of $\gamma = 1$ for the TVEAR imposes the constraint $|\lambda_{k,t}| < 1$ on the AR(P) eigenvalues within the TVEAR at all points in time. This setting ensures that the AR(P) will always be stationary/non-explosive, and so ensures aspects such as non-explosive forecasts/IRFs and the statistical validity of variance decompositions, etc. Conversely, there is nothing preventing the AR(P) coefficients of a TVOAR evolving to values that would result in explosive dynamics.

The TVOAR measurement equation is linear with respect to the state equation, and so the standard linear Kalman filter may be used for its estimation. The TVEAR measurement equation has a non-linear dependence on x_t, and so a non-linear Kalman filter is required for its estimation. I use the extended Kalman filter which, given the linear state equation, simply requires the measurement equation to be linearized using the Jacobian of $\phi(x_t) Y_{t-1}$ with respect to x_t, evaluated at the prior estimate of x_t.[10] Given that Y_{t-1} has no dependence on x_t, the Jacobian of $\phi(\lambda[x_t]) Y_{t-1}$ is best expressed as the $P \times P$ Jacobian of $\phi(\lambda[x_t])$ with respect to x_t, i.e. $\frac{\partial}{\partial x_t'}[\phi(x_t)]'$, multiplied into Y_{t-1}. As already outlined in sections 4.3.1 and 4.3.2 for NLS estimations of the PREAR and CREAR, it is feasible to calculate $\frac{\partial}{\partial x_t'}[\phi(x_t)]'$ analytically.

5 AR(P) forecasts/IRFs and components

All of the methods discussed in section 4 directly or indirectly produce estimates of ϕ and Ω_ε, and so forecasts/IRFs from an AR(P) may be obtained recursively in the usual way. That is, for point forecasts or IRFs, the prevailing data and its relevant lagged values $y_t, \ldots, y_{t-P+1}$ or a given innovation vector is used in conjunction with ϕ to obtain y_{t+1}, then $y_{t+1}, \ldots, y_{t-p+1}$ obtains y_{t+2}, etc. The associated variances for forecasts/IRFs may also be obtained recursively; e.g. see Lütkepohl (2006) pp. 36-38.[11]

However, as I show in this section, the eigensystem decomposition of the companion form of the AR(P) allows convenient closed-form expressions for forecasts/IRFs and their associated variances to be obtained as a direct function of the horizon h. These closed-form expressions in turn show how AR(P) data and forecasts/IRFs may be viewed as AR(1) and AR(2) components, which provides a useful perspective for assessing the dynamics of the AR(P).

To establish the overview above, I first introduce the companion form of the AR(P) and its associated eigensystem in section 5.1. Section 5.2 uses the companion-form expressions to derive a closed-form expression for AR(P) point forecasts/IRFs, which are the sum of AR(1) and AR(2) processes, and section 5.3 shows how the history of the time series, i.e. the data used to estimate the AR(P), may also be decomposed into those AR(1) and AR(2) components. Section 5.4 uses the AR(P) companion form expressions to derive a closed-form expressions for forecast/IRF confidence intervals, which is extended to closed-form expressions for the ergodic variances of the AR(P) and its components. Section 5.5 discusses how particular innovation vectors may be used to generate IRFs that reflect given subsets of components. The results in sections 5.1 to 5.5 are all on the basis that the eigenvalues are distinct. Section 5.6 discusses the case of repeated eigenvalues and its implications.

[10] Two alternatives are the iterated extended Kalman filter, which also requires the measurement equation to be linearized, or the unscented Kalman filter.

[11] Unless otherwise specified in particular contexts, I generally use the terminology "forecasts/IRFs" because the mechanics for generating forecasts and IRFs along with their confidence intervals is the same. The only difference, as will be discussed in section 5.5, is that IRFs use a given innovation vector while forecasts use the initial vector $[y_t, \ldots, y_{t-P+1}]'$.

5.1 AR(P) companion matrix form

The $P \times P$ companion matrix Φ associated with an AR(P) is:[12]

$$\Phi = \begin{bmatrix} \phi_1 & \cdots & \phi_{P-1} & \phi_P \\ 1 & & \mathbf{0}_\triangle & 0 \\ & \ddots & & \vdots \\ \mathbf{0}_\triangledown & & 1 & 0 \end{bmatrix} \tag{33}$$

where $\mathbf{0}_\triangledown$ and $\mathbf{0}_\triangle$ respectively denote zeros for upper- and lower-triangular elements of a matrix (in this case the $(P-1) \times (P-1)$ submatrix in Φ). Using the companion matrix, the companion form of the AR(P) is:

$$Y_t = \Phi Y_{t-1} + E_{Y,t} \tag{34}$$

where $Y_t = [y_t, \ldots, y_{t-p+1}]'$, $Y_{t-1} = [y_{t-1}, \ldots, y_{t-p}]'$, and $E_{Y,t} = [\varepsilon_t, 0, \ldots, 0]'$, which are all $P \times 1$ vectors. The variance of $E_{Y,t}$ is $\Omega_{E_Y} = \text{diag}([\Omega_\varepsilon, 0 \ldots, 0])$.

The regression form of the AR(P) may be recovered from the companion form using the $1 \times P$ vector J defined as $J = [1, 0, \ldots, 0]$. Hence, multiplying both sides of equation 34 by J recovers the regression form of the AR(P) in the last line of equation 12. Similarly, $J\Omega_{E_Y}J' = \Omega_\varepsilon$ recovers the regression form variance Ω_ε (and $\Omega_{E_Y} = J'\Omega_\varepsilon J$).

The matrix Φ may be expressed as its eigensystem decomposition, i.e.:

$$\Phi = V \Lambda V^{-1} \tag{35}$$

where V is the $P \times P$ eigenvector matrix containing the P eigenvectors (each of length P) in its columns,[13] and Λ is the $P \times P$ eigenvalue matrix containing the eigenvalues on its diagonal:

$$V = \begin{bmatrix} \lambda_1^{P-1} & \cdots & \lambda_P^{P-1} \\ \vdots & \cdots & \vdots \\ \lambda_1 & \cdots & \lambda_P \\ 1 & \cdots & 1 \end{bmatrix}; \Lambda = \begin{bmatrix} \lambda_1 & & \mathbf{0}_\triangle \\ & \ddots & \\ \mathbf{0}_\triangledown & & \lambda_P \end{bmatrix} = \text{diag}\left([\lambda_1, \ldots, \lambda_P]\right) \tag{36}$$

5.2 AR(P) closed-form forecasts/IRFs and components

From Lütkepohl (2006) p. 36, forecasts associated with an AR(P) may be expressed as:

$$\mathbb{E}_t\left[y_{t+h}\right] = J\Phi^h Y_t \tag{37}$$

where $\mathbb{E}_t\left[y_{t+h}\right]$ is the expected value, as at time t, of y_{t+h} with h the horizon in periods from t.

[12] For example, see Hamilton (1994) pp. 7-8.

[13] See Wilkinson (1965) p. 14, or Hamilton (1994) pp. 22-23. Note that V is a Vandermonde matrix, e.g. see Horn and Johnson (1991) section 6.1. Vandermonde matrices are applied in areas such as polynomial interpolation, signal processing, and control theory, and in such contexts there exist fast and accurate algorithms for inverting V. I have not used such approaches in the analysis underlying the present article, because the lag lengths are relatively short and explicit inverses are not required to calculate the products of vectors or matrices with an inverse matrix (i.e. Gaussian elimination may be used). However, explicit inversions may be computationally more efficient for longer lag lengths.

As formalized in the following proposition and proof, expressing equation 37 in its eigenvalue form provides the basis for expressing $\mathbb{E}_t[y_{t+h}]$ in closed-form as the sum of P AR(1) components, each associated with one of the AR(P) eigenvalues.

Proposition 4 *If all eigenvalues are distinct, $\mathbb{E}_t[y_{t+h}]$ is a sum of P AR(1) forecasts/IRFs, i.e.:*

$$\mathbb{E}_t[y_{t+h}] = \sum_{k=1}^{P} \lambda_k^h X_{k,t} \tag{38}$$

with the eigenvalues $[\lambda_1, \ldots, \lambda_P]$ providing the AR(1) coefficients, and the $P \times 1$ vector $X_t = \Lambda^{P-1} V^{-1} Y_t$ providing the initial values for each AR(1).

Proof. Substituting $\Phi = V \Lambda V^{-1}$ from equation 35 into $\mathbb{E}_t[y_{t+h}] = J \Phi^h Y_t$ and using the form of the eigenvector and eigenvalue matrices in equation 36 gives the result:

$$\begin{aligned}
\mathbb{E}_t[y_{t+h}] &= J\left(V\Lambda V^{-1}\right)^h Y_t \\
&= JV\Lambda^h V^{-1} Y_t \\
&= \left[\lambda_1^{P-1}, \ldots, \lambda_P^{P-1}\right] \Lambda^{h-P+1} \left(\Lambda^{P-1} V^{-1} Y_t\right) \\
&= \left[\lambda_1^{P-1}, \ldots, \lambda_P^{P-1}\right] \Lambda^{h-P+1} X_t \\
&= \left[\lambda_1^{h}, \ldots, \lambda_P^{h}\right] X_t \\
&= [1, \ldots, 1]\, \Lambda^h X_t
\end{aligned} \tag{39}$$

where $\left[\lambda_1^h, \ldots, \lambda_P^h\right]$ is a $1 \times P$ vector, and $X_t = \Lambda^{P-1} V^{-1} Y_t = [X_{1,t}, \ldots, X_{P,t}]'$ is a $P \times 1$ vector. Evaluating the inner product $\left[\lambda_1^h, \ldots, \lambda_P^h\right] X_t$ or the equivalent $[1, \ldots, 1]\, \Lambda^h X_t$ gives the summation form in equation 38. ■

Unless the eigenvalues have been constrained to be real, Λ will generally include real eigenvalues and pairs of complex conjugate eigenvalues. In the case of a real eigenvalue λ_k, its associated component $\lambda_k^h X_{k,t}$ will be a real AR(1) process. Complex eigenvalues could be accommodated individually as complex AR(1) models, e.g. see Sekita, Kurita, and Otsu (1992), but it is more convenient to remain in the real domain by combining the contribution of complex conjugate components $\lambda_k^h X_{k,t}$ and $\lambda_{k+1}^h X_{k+1,t}$ to $\mathbb{E}_t[y_{t+h}]$ into a real AR(2), as in the following proposition.

Proposition 5 *A pair of components $\lambda_k^h X_{k,t} + \lambda_{k+1}^h X_{k+1,t}$ associated with the complex conjugate eigenvalues $(\lambda_k, \lambda_{k+1})$ will contribute the following AR(2) forecast/IRF component to $\mathbb{E}_t[y_{t+h}]$:*

$$\lambda_k^h X_{k,t} + \lambda_{k+1}^h X_{k+1,t} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_k^* & \phi_{k+1}^* \\ 1 & 0 \end{bmatrix}^h \begin{bmatrix} X_{k,t}^* \\ X_{k,t-1}^* \end{bmatrix} \tag{40}$$

where $\left[X_{k,t}^*, X_{k,t-1}^*\right]' = \left[2\,\mathrm{Re}\left(\lambda_k X_{k,t}\right), 2\,\mathrm{Re}\left(X_{k,t}\right)\right]$, and the pair of AR(2) coefficients is $\left(\phi_k^*, \phi_{k+1}^*\right) = \left(\lambda_k + \lambda_{k+1}, -\lambda_k \lambda_{k+1}\right)$. The latter may also be expressed as $\left(\phi_k^*, \phi_{k+1}^*\right) = \left(2\,\mathrm{Re}\left(\lambda_k\right), -\left|\lambda_k\right|^2\right)$.

Proof. See sections C.1 and C.2 of appendix C. ■

Like for the AR(2) in its own right (see section A.3 of appendix A), using the polar form for a complex conjugate pair of eigenvalues, i.e. $(\lambda_k, \lambda_{k+1}) = [r\exp(i\theta_k), r\exp(-i\theta_k)]$ where $\theta = \cos^{-1}(\text{real}[\lambda_k] / |\lambda_k|)$, results in expressions for the AR(2) components based on trigonometric functions, i.e. $r\sin(h\theta_k)$ and $r\cos(h\theta_k)$. Section C.3 in appendix C contains further details, and the trigonometric perspective clearly shows the oscillatory nature of the contributions to forecasts/IRFs from AR(2) components.

In summary then, the forecasts/IRFs for an AR(P) with distinct eigenvalues may be expressed as a sum of real AR(1) and AR(2) processes, with each AR(1) associated with one of the real eigenvalues and each AR(2) associated with one of the pairs of complex conjugate eigenvalues, i.e.:

$$\mathbb{E}_t[y_{t+h}] = \sum_k^{\text{real } \lambda_k} \lambda_k^h X_{k,t} + \sum_k^{\text{complex } \lambda_j} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_k^* & \phi_{k+1}^* \\ 1 & 0 \end{bmatrix}^h \begin{bmatrix} X_{k,t}^* \\ X_{k,t-1}^* \end{bmatrix} \tag{41}$$

where "real λ_k" denotes the set of real eigenvalues, and "complex λ_j" denotes sets of complex conjugate pairs of eigenvalues, i.e. each component j represents the combined contribution of the eigenvalue pair $(\lambda_k, \lambda_{k+1}) = (\lambda_k, \overline{\lambda_k})$, where $k = 2j-1$ and $k+1 = 2j$.

5.3 Historical component decomposition

The framework for decomposing $\mathbb{E}_t[y_{t+h}]$ into components may also be applied to the historical time series y_t, i.e. the data used to estimate the AR(P), by simply setting the horizon to $h = 0$, given that $y_t = \mathbb{E}_t[y_{t+0}]$. Using the real AR(1) and AR(2) processes in equation 41, the summation is:[14]

$$\begin{aligned} y_t &= \sum_k^{\text{real } \lambda_k} X_{k,t} + \sum_j^{\text{complex } \lambda_j} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_{k,t}^* \\ X_{k,t-1}^* \end{bmatrix} \\ &= \sum_k^{\text{real } \lambda_k} X_{k,t} + \sum_j^{\text{complex } \lambda_j} X_{k,t}^* \end{aligned} \tag{42}$$

The historical components may be interpreted in two senses. First, if an AR(P) forecast from time t was made, then $X_{k,t}$ or $X_{k,t}^*$ would be the starting value for the AR(1) or AR(2) component of the AR(P) forecast.[15] Second, as shown in the following two propositions, each time series $X_{k,t}$ or $X_{k,t}^*$ is effectively "decomposed data", i.e. a time series that if used for estimation in its own right would produce the component AR(1) or AR(2) model. Specifically, Proposition 6 establishes that an AR(1) estimation using the time series $X_{k,t}$ would result in the eigenvalue λ_k as the coefficient, although the AR(1) could be real or complex depending on λ_k. Proposition 7 establishes that $X_{k,t}^*$ would result in the coefficient pair (ϕ_k^*, ϕ_{k+1}^*) for the real AR(2) that is associated with a complex conjugate pair of eigenvalues $(\lambda_k, \lambda_{k+1})$.

[14] The data y_t could also be decomposed into the real and complex AR(1) components from equation 38, in which case the summation is $y_t = [1, \ldots, 1] X_t = \sum_{k=1}^P X_{k,t}$, and the right-hand side is real due to the cancellation of the imaginary parts of the complex conjugate components.

[15] The forecasts will obviously be in-sample, because the eigenvalues used to obtain $X_{k,t}$ or $X_{k,t}^*$ are estimated from the full sample, directly via the EAR or indirectly via the OAR coefficients.

Proposition 6 *An AR(1) model estimated with the decomposed data series $X_{k,t}$ will give the eigenvalue λ_k as the AR(1) coefficient.*

Proof. Beginning with the companion form for the AR(P) and using the eigensystem decomposition for Φ, i.e. $\Phi = V\Lambda V^{-1}$, gives the following result:

$$\begin{aligned}
Y_t &= \Phi Y_{t-1} + E_{Y,t} \\
Y_t &= V\Lambda V^{-1} Y_{t-1} + E_{Y,t} \\
V^{-1} Y_t &= \Lambda V^{-1} Y_{t-1} + V^{-1} E_{Y,t} \\
\Lambda^{P-1} V^{-1} Y_t &= \Lambda \Lambda^{P-1} V^{-1} Y_{t-1} + \Lambda^{P-1} V^{-1} E_{Y,t} \\
X_t &= \Lambda X_{t-1} + E_{X,t}
\end{aligned} \tag{43}$$

where the fourth line uses the result that $\Lambda\Lambda^{P-1} = \Lambda^{P-1}\Lambda$, because Λ is diagonal, and $E_{X,t} = \Lambda^{P-1}V^{-1}E_{Y,t}$. Equation 43 in full matrix form is:

$$\begin{bmatrix} X_{1,t} \\ \vdots \\ X_{P,t} \end{bmatrix} = \begin{bmatrix} \lambda_1 X_{1,t-1} & & \mathbf{0}_{\triangle} \\ & \ddots & \\ \mathbf{0}_{\triangledown} & & \lambda_P X_{P,t-1} \end{bmatrix} + \begin{bmatrix} E_{X,1,t} \\ \vdots \\ E_{X,P,t} \end{bmatrix} \tag{44}$$

and each line is an AR(1) in regression form, i.e.:

$$X_{k,t} = \lambda_k X_{k,t-1} + E_{X,k,t} \tag{45}$$

■

Proposition 7 *Estimating an AR(2) model with the decomposed data series $X^*_{k,t}$ will return AR(2) coefficients associated with the complex conjugate eigenvalue pair* $(\lambda_k, \lambda_{k+1})$.

Proof. See section C.4 of appendix C. ■

5.4 Closed-form forecasts/IRFs and variances

Confidence intervals around the point forecasts and IRFs for a given horizon may be obtained from the forecast error variance (FEV) associated with that horizon. Following Lütkepohl (2006) p.38, eq. 2.2.11, the FEV for horizon H is obtained by summing the contributions from the moving-average representation, i.e.:[16]

$$\Omega_y(H) = \sum_{h=0}^{H-1} J\Phi^h \Omega_{E_Y} \left(\Phi^h\right)' J' \tag{46}$$

where H now represents the forecast horizon, given that I have retained h to denote the horizon for each period up to H within the FEV expression.

[16]Some alignment of my notation with Lütkepohl (2006) is necessary to show the equivalance of my expression in equation 46 and equation 2.2.11 in Lütkepohl (2006); see section D.1 of appendix D for details. Section D.1 also details how equation 46 is derived, which parallels the exposition in Lütkepohl (2006) section 2.2.2.

As noted in Lütkepohl (2006), the summation is usually done recursively, i.e. setting $\Omega_y(1) = J\Omega_{E_Y}J' = \Omega_\varepsilon$ as obtained with $H = 1$ in equation 46, and using the following recursive relationship for subsequent horizons:

$$\Omega_y(H) = \Omega_y(H) + J\Phi^{H-1}\Omega_{E_Y}\left(\Phi^{H-1}\right)' J' \tag{47}$$

The eigensystem representation for an AR(P) provides a basis for deriving a closed-form expression for the finite sum in equation 46, as outlined in Proposition 8 below. The proof is provided in section D.2 of appendix D, but the intuition is that using $\Phi = V\Lambda V^{-1}$ allows each term of the FEV summation in equation 46 to be expressed as $J\Phi^h J'\Omega_\varepsilon J\left(\Phi^h\right)' J' = JV\Lambda^h V^{-1}J'\Omega_\varepsilon J\left(V\Lambda^h V^{-1}\right)^\dagger J'$, and then the sum of each element in the matrix $\Lambda^h V^{-1}J'\Omega_\varepsilon J\left(V^{-1}\right)^\dagger \Lambda^h$ may be expressed as a closed-form geometric summation of a scalar. Note that the Hermitian transpose "†", i.e. $U_{i,j} = \overline{U_{j,i}}$, allows for complex conjugate elements.

Proposition 8 *The FEV, $\Omega_y(H)$, for a given horizon H may be obtained directly by calculating each of the (i,j) elements of $\Omega_X(H)$ as:*

$$[\Omega_X(H)]_{ij} = \Omega_{E_X,ij}\frac{1-\left(\lambda_i\overline{\lambda_j}\right)^H}{1-\lambda_i\overline{\lambda_j}} \tag{48}$$

where $\Omega_{E_X} = V_X^{-1}\Omega_{E_Y}\left(V_X^{-1}\right)^\dagger$ is the $P\times P$ covariance matrix for $E_{X,t}$, with $V_X = V\Lambda^{1-P}$, and then using the resulting $\Omega_X(H)$ in the expression:

$$\Omega_y(H) = JV_X\Omega_X(H)V_X^\dagger J' \tag{49}$$

Proof. See section D.2 of appendix D. ■

I am not aware of this or any other closed-form expression for AR(P) FEVs in the literature. This may be because FEVs are typically calculated for every horizon out to longest horizon of interest (e.g. to produce figures of forecasts/IRFs with confidence intervals), in which case the recursive expression is convenient and computationally efficient. But if FEVs are only required for one or several horizons, then the closed-form expression offers a computationally efficient means of doing so, given the FEV for a particular horizon/s may be calculated without the FEVs for intervening horizons.

The closed-form FEV expressed in terms of the AR(P) eigenvalues in Proposition 8 may also be used to obtain a closed-form expression for the ergodic variance of the AR(P), simply by taking the limit as $H \to \infty$.

Proposition 9 *The ergodic variance $\Omega_y(\infty)$ for an AR(P) may be obtained by calculating each (i,j) element of $\Omega_X(\infty)$ as:*

$$[\Omega_X(\infty)]_{ij} = \Omega_{E_X,ij}\frac{1}{1-\lambda_i\overline{\lambda_j}} \tag{50}$$

and then using the resulting $\Omega_X(\infty)$ in the expression:

$$\Omega_y(\infty) = JV_X\Omega_X(\infty)\left(V_X\right)^\dagger J' \tag{51}$$

Proof. Using equation 49 and taking the limit as $H \to \infty$.gives:

$$\begin{aligned}\Omega_y(\infty) &= \lim_{H\to\infty}\left(JV_X\Omega_X(H)V_X^{\dagger}J'\right)\\ &= JV_X\Omega_X(\infty)V_X^{\dagger}J'\end{aligned}$$

and then the elements of $\Omega_X(\infty)$ are calculated by taking the limit as $H \to \infty$ of equation 48:

$$\begin{aligned}\Omega_X(\infty) &= \lim_{H\to\infty}\left(\Omega_{E_X,ij}\frac{1-\left(\lambda_i\overline{\lambda_j}\right)^H}{1-\lambda_i\overline{\lambda_j}}\right)\\ &= \Omega_{E_X,ij}\frac{1}{1-\lambda_i\overline{\lambda_j}} \qquad (52)\end{aligned}$$

■

Again, I am not aware of this particular result in the literature, but it is within the class of other computationally efficient methods for obtaining the ergodic variance or, equivalently, solving the discrete Lyanpunov equation. Section D.3 of appendix D contains further discussion related to Proposition 9 and its context in the literature.

The ergodic variance result above also suggests a means of assessing the importance of each of the AR(1) or AR(2) components to the dynamics of the AR(P). The following proposition provides the basis for such an assessment.

Proposition 10 *The ergodic variances for each of the AR*(1) *components in Proposition 4 are:*

$$\Omega_{E_{X,kk}}(\infty) = \frac{\Omega_{E_{X,kk}}}{1-\lambda_k\overline{\lambda_k}} = \frac{\Omega_{E_{X,kk}}}{1-|\lambda_k|^2} \qquad (53)$$

which are also the diagonal elements of $\Omega_X(\infty)$*.*

Proof. The regression form of the AR(1) component k for an AR(P) is $X_{k,t} = \lambda_k X_{k,t-1} + E_{X,k,t}$, which arises from each line of $X_t = \Lambda X_{t-1} + E_{X,t}$, where $X_t = \Lambda^{P-1}V^{-1}Y_t$ and $E_{X,t} = \Lambda^{P-1}V^{-1}E_{Y,t}$. Defining the variance for $E_{X,k,t}$ directly as:

$$\Omega_{E_{X,k}} = \mathbb{E}\left(E_{X,k,t}E_{X,k,t}^{\dagger}\right) \qquad (54)$$

then, from equation 52, the AR(1) component will have the following ergodic variance:

$$\Omega_{E_{X,k}}(\infty) = \Omega_{E_{X,k}}\frac{1}{1-\lambda_k\overline{\lambda_k}} = \frac{\Omega_{E_{X,k}}}{1-|\lambda_k|^2} \qquad (55)$$

Also from equation 52, the (k,k) element of $\Omega_X(\infty)$ is:

$$[\Omega_X(\infty)]_{kk} = \Omega_{E_X,kk}\frac{1}{1-\lambda_k\overline{\lambda_k}} \qquad (56)$$

which shows that $\Omega_{E_{X,k}}(\infty) = [\Omega_X(\infty)]_{kk}$. ■

Note that $\Omega_{E_{X,kk}}(\infty)$ is necessarily real, because $\Omega_{E_{X,k}}$ and $1-|\lambda_k|^2$ are real. Therefore, Proposition 10 applies to AR(1) and AR(2) components, or specifically to both real AR(1) components and each complex AR(1) component within a pair of complex conjugate components that form an AR(2). Based on Proposition 10, the ergodic variances for each of the AR(1) and AR(2) components in section 10 may be used to show their direct contribution to the ergodic variance of the AR(P), which provides a quantitative measure of their dynamic importance. I illustrate such a comparison in the empirical applications within section 6.

5.5 AR(P) closed-form IRFs and IRF components

An IRF for an AR(P) is simply a forecast associated with a given innovation vector. I will denote the innovation vector as $Y_{0,t}$, and its associated IRF as $\mathbb{E}_t\left[y_{t+h}|Y_{0,t}\right]$. The innovation vector $Y_{0,t}$ may be transformed into the component form, i.e. $X_{0,t}=\Lambda^{P-1}V^{-1}Y_{0,t}$, in which case the IRF may also be represented in component form, i.e. $\mathbb{E}_t\left[X_{t+h}|X_{0,t}\right]=\Lambda^{P-1}V^{-1}\mathbb{E}_t\left[Y_{t+h}|Y_{0,t}\right]$.

An IRF is typically calculated for a contemporaneous innovation, which corresponds to a non-zero value in just the first element of $Y_{0,t}$, e.g. $Y_{0,t}=[1,0,\ldots,0]'$ for a unit innovation. In this case, the vector $X_{0,t}=\Lambda^{P-1}V^{-1}[1,0,\ldots,0]'$ will generally have P non-zero elements, and so the closed-form IRF will be a sum over P non-zero functions $\lambda_k^h X_{0,k,t}$ as follows:

$$\mathbb{E}_t\left[y_{t+h}|Y_{0,t}\right]=\sum_{k=1}^{P}\lambda_k^h X_{0,k,t} \tag{57}$$

If non-contemporaneous values in $Y_{0,t}$ are also allowed to be non-zero, which would be in the context to be discussed further below, then it is possible to deliver IRFs that have just the dynamics associated with particular eigenvalue components. For example, an IRF that is only a function of the first eigenvalue component, i.e. $\mathbb{E}_t\left[y_{t+h}|Y_{0,t}\right]=\lambda_1^h X_{1,t}$, would be delivered with the vector $X_{0,t}=\Lambda^{P-1}[1,0,\ldots,0]'$, which in turn would be obtained with an innovation $Y_{0,t}$ set equal to the first eigenvector, i.e.:

$$\begin{aligned}
X_{0,t} &= \Lambda^{P-1}[1,0,\ldots,0]' \\
\Lambda^{P-1}V^{-1}Y_{0,t} &= \Lambda^{P-1}[1,0,\ldots,0]' \\
Y_{0,t} &= V[1,0,\ldots,0]' \\
&= V_1
\end{aligned} \tag{58}$$

Similarly, using the innovation $Y_{0,t}=V_k$ will result in $X_{0,t}=\Lambda^{P-1}\mathrm{e}_k$ (where e_k is a $P\times 1$ vector with an entry of 1 in element k and entries of zero otherwise), and therefore the IRF $\mathbb{E}_t\left[y_{t+h}|Y_{0,t}\right]=\lambda_k^h X_{k,t}$. An innovation $Y_{0,t}=[V_k+V_{k+1}]=2\,\mathrm{Re}\left(V_k\right)$ where V_k and V_{k+1} are associated with a complex conjugate eigenvalue pair $(\lambda_k,\lambda_{k+1})$ will result in $X_{0,t}=\Lambda^{P-1}\left(\mathrm{e}_k+\mathrm{e}_{k+1}\right)$. The IRF is therefore $\mathbb{E}_t\left[y_{t+h}|Y_{0,t}\right]=\lambda_k^h X_{k,t}+\lambda_{k+1}^h X_{k+1,t}$ which, from Proposition 5, is an IRF for just the AR(2) component of the AR(P) associated with $(\lambda_k,\lambda_{k+1})$ or equivalently $\left(\phi_k^*,\phi_{k+1}^*\right)$. More generally, an innovation $Y_{0,t}=VX_{0,t}$ calculated from $X_{0,t}$ with selected non-zero elements will result in $Y_{0,t}$ being a linear combination of the selected eigenvectors, so $Y_{0,t}$ will in turn produce an IRF $\mathbb{E}_t\left[y_{t+h}|Y_{0,t}\right]$ that is a linear combination of the eigenvalue components associated with the non-zero elements in $X_{0,t}$.

Of course, the example of $Y_{0,t}=V_k$ and its variants in the previous paragraph are no longer contemporaneous innovations. That is, the eigenvector associated with a given eigenvalue λ_k is $V_k = \left[\lambda_k^{P-1}, \lambda_k^{P-2}, \ldots, \lambda_k, 1\right]'$, and so $Y_{0,t} = \left[\lambda_k^{P-1}, \lambda_k^{P-2}, \ldots, \lambda_k, 1\right]'$ would contain the contemporaneous value $y_{0,t} = \lambda_k^{P-1}$, and the respective values of $y_{0,t-1} = \lambda_k^{P-2}$, $y_{0,t-2} = \lambda_k^{P-3}$, ..., $y_{0,t-P+2} = \lambda_k$, and $y_{0,t-P+1} = 1$. The context in which an innovation vector would deliver an IRF associated with a single eigenvalue component is therefore a specification with a contemporaneous value of 1 at time t, and an ex-ante sequence of future innovations, i.e. λ_k at time $t+1$, λ_k^2 at time $t+2$, etc. until λ_k^{P-1} at time $t+P-1$. To distinguish this from $Y_{0,t}$, I denote this ex-ante innovation vector as $Y_{P,t} = \left[\lambda_k^{P-1}, \lambda_k^{P-2}, \ldots, \lambda_k, 1\right]'$, and the associated IRF as $\mathbb{E}_t\left[y_{t+h}|Y_{P,t}\right]$. More generally, analogous to the discussion at the end of the previous paragraph, $Y_{P,t}$ suitably defined with a linear combination of eigenvectors, via $Y_{P,t} = VX_{P,t}$ with $X_{P,t}$ containing selected non-zero elements, would produce an IRF $\mathbb{E}_t\left[y_{t+h}|Y_{P,t}\right]$ that is a linear combination of the eigenvalue components associated with the non-zero elements in $X_{P,t}$.

While perhaps unusual in a pure time series context, an ex-ante innovation specification like $Y_{P,t}$ already has a precedent in Dynamic Stochastic General Equilibrium modelling, e.g. Carrillo, Feve, and Matheron (2011) uses an ex-ante innovation specification to allow for more persistent shocks to monetary policy. That example also helps to clarify the meaning of an ex-ante innovation introduced above, i.e. $Y_{P,t}$ should not be viewed as a series of unanticipated shocks, but rather a single shock that contains a contemporaneous intervention along with a series of pre-announced future interventions.

5.6 Repeated eigenvalues

The assumption of distinct eigenvalues in sections 5.1 to 5.5 covers the results of all empirical OAR estimations and also estimations of EARs that are specified with distinct eigenvalues. Specifically, it would be rare to obtain even approximately equal eigenvalues from such estimations and, in any case, machine precision underlying the numerical processes for estimation will ensure that eigenvalues are not repeated.

But repeated eigenvalues may be imposed on an AR(P) estimation using the EAR framework, such as the single pair of repeated eigenvalues mentioned in section 4.3.3. In general, an EAR could be specified to include more than two repeats of a single eigenvalue, or several groups of repeated eigenvalues. Section E.1 of appendix E provides an overview of how these general cases may be accommodated, essentially by incorporating the appropriate Jordan block/s in the eigenvalue matrix Λ and making the associated adjustments to the corresponding eigenvectors in V.

Repeated eigenvalues therefore produce components in the closed-form forecast/IRF expressions with functional forms outside of those already presented for the AR(1) and AR(2) processes. For example, as detailed in Section E.2 of appendix E, the EAR specifications with a single pair of repeated eigenvalues, i.e. $\lambda_1 = \lambda_2$ (so both must be real), that I apply in section 6 will contribute to $\mathbb{E}_t\left[y_{t+h}\right]$ the following forecast/IRF component:

$$\lambda_1^h X_{1,t} + (h + P - 1)\,\lambda_1^{h-1} X_{2,t} \tag{59}$$

where $X_{1,t} = \left[\Lambda^{P-1}V^{-1}Y_t\right]_1$ and $X_{2,t} = \left[\Lambda^{P-1}V^{-1}Y_t\right]_2$ are respectively elements 1 and 2 of the $P \times 1$ vector $X_t = \Lambda^{P-1}V^{-1}Y_t$. The historical decompositions, FEVs, and ergodic variances also have adjustments associated with the repeated eigenvalue pair.

The rest of Λ and V remain in the form already discussed for distinct eigenvalues in section 5.1, and so the associated components in the forecast/IRF expressions, etc., remain as presented in sections 5.2 to 5.5.

6 Empirical applications

In this section, I apply the EAR framework to mean-adjusted quarterly United States 3-month Treasury bill (US Tbill) rate data. In section 6.1, I estimate and provide results for two OARs and then a set of EARs subject to the variety of eigenvalue constraints mentioned in the introduction. Section 6.2 provides an example of historical and forecast decompositions of an AR(P) into eigenvalue components. Section 6.3 contains the results of the TVEAR specified in section 4.4.

Before proceeding, it is important to make several aspects clear. First and foremost, for the empirical examples in sections 6.1 and 6.2, I have used a selected sample period and have estimated the models and forecast with them in particular ways to most clearly illustrate the dynamics associated with the eigenvalues within the EAR framework. Specifically, the sample period I have chosen is from Jun-1947 to Mar-1981 (so $P+T=136$), I have imposed the lag lengths of $P=4$ and 5 (which also ensures that the differences within the two groups of models are due only to the eigenvalue constraints), and I have used a forecast horizon of 20 years. These choices provide results that are visually apparent in the figures, both for the EAR estimations with eigenvalue constraints in section 6.1, and the component decomposition example in section 6.2. The long horizons for forecasts also allow the forecasts in the figures to serve as IRF illustrations. Specifically, in figure 3 of section 6.1, the largest eigenvalue components for the given models dominate the forecasts for longer horizons, so the latter are effectively the long-horizon IRF $\mathbb{E}_t\left[y_{t+h}|Y_{P,t}\right]$ associated with the innovation vector $Y_{P,t}=\lambda_1^{P-1}V_1=\lambda_1^3 V_1$. Figure 4 in section 6.2 contains the forecast components associated with individual eigenvalue components, which are IRFs $\mathbb{E}_t\left[y_{t+h}|Y_{P,t}\right]$ with innovation vectors $Y_{P,t}=X_{k,t}V\,\mathrm{e}_k$ for a real AR(1) eigenvector component or $Y_{P,t}=X_{k,t}V\left(\mathrm{e}_k+\mathrm{e}_{k+1}\right)$ for a real AR(2) component associated with a pair of complex conjungate eigenvalues.

My choices underlying these examples should obviously not be taken as an advocation to model and forecast the US Tbill rate in any of the particular ways shown; indeed, the log-likelihood ratios indicate that some models are clearly rejected by the data,[17] and the nature of some forecasts in the figures also show that they would be inappropriate when forecasting interest rates. In practice, for this variable or any other, a researcher would specify, estimate, and evaluate their own AR model according to their particular requirements, including pre-testing to select the appropriate lag length and using diagnostics and judgement to ensure that the model was appropriate.

6.1 AR(P) model estimations

Table 1 contains the results for estimating AR(P) models with $P=4$ and $P=5$, respectively, first via the OAR in models 1 and 11, and then via the EAR in models 2-10 and 12-20. Within each of the EAR estimations, models 2-7 and 12-17 are CREARs (i.e. eigenvalues may be complex or real), and models 8-10 and 18-20 are PREARs (i.e. with only real eigenvalues). I report results to four decimal places for the largest eigenvalues in each model to clearly show the effect of constraints, as discussed further below.

[17] The critical chi-squared values of 10% 2.71, 5% 3.84, 2.5%, 5.02, and 1% 6.63 provided in table 1 are from the standard chi-squared distribution with one degree of freedom (to allow for a single constraint). These values are should be treated as indicative only because they will not apply to the non-stationary models (which will have non-standard distributions).

Table 1: Estimated AR(4) and AR(5) models via OAR and EAR framework

Model and eigenvalue constraints[(1)]		Coefficients ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	σ	LLR[(2)]	Eigenvalues λ_1	λ_2	λ_3	λ_4	λ_5
1.	None, OAR(4)	0.77	–0.21	0.48	–0.03	·	1.00	0.00	1.0103	–0.15±0.67i[(3)]		0.06	·
	standard errors	0.09	0.11	0.11	0.10		0.09		0.0209	0.10±0.06i[(3)]		0.21	·
2.	$\lvert\lambda_k\rvert < 2$	0.77	–0.21	0.48	–0.03	·	1.00	0.00	1.0103	–0.15±0.67i		0.06	·
3.	$\lvert\lambda_k\rvert < 1 + \frac{1}{T}$	0.77	–0.21	0.48	–0.03	·	1.00	0.01	1.0074	–0.15±0.67i		0.07	·
4.	$\lvert\lambda_k\rvert < 1$	0.78	–0.22	0.48	–0.04	·	1.00	0.12	1.0000	–0.15±0.67i		0.08	·
5.	$\lvert\lambda_k\rvert < 0.95$	0.82	–0.24	0.47	–0.13	·	1.02	3.08	0.9500	–0.20±0.66i		0.28	·
6.	$\lvert\lambda_{1,2}\rvert = 1$, $\lvert\lambda_k\rvert < 1$	0.72	–0.49	1.13	–0.36	·	1.13	16.04	–0.3213±0.9470i		1.00	0.36	·
7.	$\lambda_1 = \lambda_2$, $\lvert\lambda_k\rvert < 1$	0.98	–0.22	0.53	–0.37	·	1.07	9.49	0.8024		–0.31±0.70i		·
8.	$\lambda_k < 1$	0.95	–0.00	0.00	–0.00	·	1.11	13.79	0.9545	0.00	0.00	0.00	·
9.	$\lambda_1 = 1$, $\lambda_k < 1$	1.00	–0.00	0.00	–0.00	·	1.12	14.62	1.0000	0.00	0.00	0.00	·
10.	$\lambda_1 = \lambda_2$, $\lambda_k < 1$	1.16	–0.34	0.00	–0.00	·	1.23	27.85	0.5811		0.00	0.00	·
11.	None, OAR(5)	0.75	–0.17	0.48	0.38	–0.46	0.96	0.00	0.9763	–0.10±0.95i		–0.73	0.71
	$X_{k,t=\text{Mar-1981}}$								8.45	0.54±1.92i		0.27	-0.61
	ergodic variances								18.88	0.46		0.02	0.35
12.-14.	Respective constraints as for models 2-4, and all results are as for the OAR(5), model 11												
15.	$\lvert\lambda_k\rvert < 0.90$	0.80	–0.13	0.42	0.26	–0.41	0.97	1.59	–0.1106±0.8932i		0.90	0.82	-0.69
16.	$\lvert\lambda_{1,2}\rvert = 1$, $\lvert\lambda_k\rvert < 1$	0.74	–0.20	0.51	0.48	–0.56	0.97	0.40	–0.1038±0.9946i		0.97	–0.77	0.76
17	$\lambda_1 = \lambda_2$, $\lvert\lambda_k\rvert < 1$	0.78	–0.17	0.48	0.42	–0.56	0.97	0.61	0.8772		–0.11±0.97i		-0.76
18.	$\lambda_k < 1$	0.95	–0.00	0.00	–0.00	0.00	1.11	18.93	0.9541	0.00	0.00	0.00	0.00
19.	$\lambda_1 = 1$, $\lambda_k < 1$	1.00	–0.00	0.00	–0.00	0.00	1.12	19.76	1.0000	0.00	0.00	0.00	0.00
20.	$\lambda_1 = \lambda_2$, $\lambda_k < 1$	1.16	–0.34	0.00	–0.00	0.00	1.24	32.79	0.5800		0.00	0.00	0.00

Notes: (1) Models 1 and 11 are the respective OAR estimates with P=4 and P=5, and the eigenvalues are calculated from the coefficient estimates. Models 2 to 10 and 12 to 20 are EAR estimates with P=4 and P=5, respectively, subject to the given equality or inequality on the eigenvalues, and the coefficients are calculated from the eigenvalue estimates. (2) The respective OAR(4) and OAR(5) log-likelihoods are –187.2986 and -185.8434, and log-likelihoods ratios indicate the significance of the constraint, with the critical chi-squared values of 10% 2.71, 5% 3.84, 2.5%, 5.02, and 1% 6.63 (also see footnote 16 in the text). (3) The polar forms for these entries are provided in the discussion of the OAR(4) results in the text.

Figure 3 plots selected forecasts from the $P = 4$ suite of model estimates to illustrate how the eigenvalue constraints in the estimated EAR models affect the model dynamics.

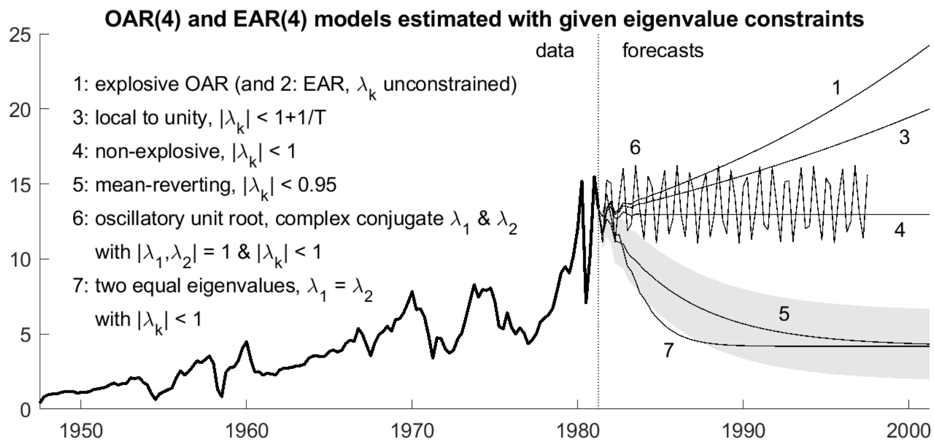

Figure 3: Forecasts for the AR(4) models 1 to 7 from table 1. The shaded area is ±1 standard deviation confidence interval around the forecast from model 5.

For $P = 4$, the OAR coefficients $[\phi_1, \ldots, \phi_4]$ and their standard errors are obtained from the estimation via OLS. The point estimates of the associated eigenvalues $[\lambda_1, \ldots, \lambda_4]$ are calculated from the eigensystem decomposition of the OAR companion matrix, and the eigenvalue standard errors are discussed in the following paragraph. The largest eigenvalue $\lambda_1 = 1.0103$ indicates that the OAR(4) is explosive, which is also clearly evident in the associated forecasts/IRFs in figure 3. The other eigenvalues for the OAR(4) have magnitudes less than 1, with λ_2 and λ_3 a complex conjugate pair ($|\lambda_2| = 0.96$), and λ_4 real.

The EAR models are all estimated via their eigenvalues, as outlined in section 4.3 using the various restrictions as given in table 1, and the coefficients are calculated via convolution of the eigenvalue estimates, as in section 3. Model 2 shows that the CREAR replicates the OAR when the inequality constraint for the eigenvalue magnitude is set large enough to not bind, i.e. $|\lambda_k| < 2$ in this example. Hence, I use model 2 as an example of calculating the eigenvalue standard errors, via the Hessian, for the unconstrained EAR and so the OAR in model 1.[18] The 0.10±0.06i entry for (λ_2, λ_3) gives the respective standard errors for the real and imaginary components. In polar form, (λ_2, λ_3) are $r \exp(\pm\theta i) = 0.68 \times \exp(\pm 1.79i)$, with respective standard errors for r and θ of 0.07 and 0.14.

The local-to-unity inequality constraint for model 3 mildly restricts the largest eigenvalue from an explosive to a mildly explosive value. Similarly, the unity inequality constraint for model 4 further restricts the largest eigenvalue to be non-explosive, and model 5 imposes a more material restriction so that the resulting model is obviously mean-reverting. These respective properties are evident in the forecast/IRF plots of models 3 to 5 in figure 3. Table 1 shows that the magnitudes of the largest estimated eigenvalues

[18] Specifically, I numerically calculate the Hessian matrix of the log-likelihood function with respect to the eigenvalue and variance estimates, i.e. $\mathbb{H} = \frac{\partial \log(\mathcal{L}[\lambda,\Omega])}{\partial[\lambda,\Omega]' \partial[\lambda,\Omega]}$. The standard errors are then $\text{diag}\left(-\mathbb{H}^{-1}\right)$. The $[\lambda, \Omega]$ standard errors for the remaining models are obtained in the same way but I have not reported these in table 1 to save space. Also note that the Hessian and hence the standard errors can only be calculated for the unconstrained subset of eigenvalues (and Ω). An eigenvalue subject to a binding inequality constraint will have very large standard errors because the corresponding row and column in the Hessian will be near zero, making the Hessian near-singular.

for models 3 to 5 essentially equal the respective constraints, i.e. $\gamma = 1 + 1/T = 1.074$, 1, and 0.95. Also included for the model 5 forecast results is the ± 1 standard deviation range obtained from the closed-form FEV expression in section 5.4.

A further point of note on models 3 to 5 is that, as the eigenvalue inequality constraint is progressively tightened, the changes in the AR(4) coefficients are small and not proportionally linear. This aspect highlights the point from the introduction that eigenvalue constraints could not be achieved via an OAR with linear constraints on its coefficients.

Models 6 and 7 are examples of applying more arbitrary eigenvalue constraints on the estimation. Hence, model 6 uses a complex conjugate eigenvalue pair with an eigenvalue magnitude equality constraint of 1 to impose an oscillatory unit root. Figure 3 shows the forecast/IRF result of non-decaying oscillations, with a wavelength of 3.31 quarters.[19] Such a constraint would obviously not be used for forecasting interest rates, but it does present a method for modelling seasonality. Model 7 constrains the first two estimated eigenvalues to be equal, which illustrates an avenue for parameter reduction/model selection.[20]

The PREAR results in models 8 and 9 show that imposing the constraint of only real eigenvalues causes the models to default to an AR(1), so $\phi_1 = \lambda_1$ and the remaining eigenvalue estimates are essentially zero. While the model produced is minimal, the PREAR nevertheless works as intended to avoid pronounced oscillatory dynamics, particularly in the AR(5) example discussed below and in section 6.2. The constraint of repeated eigenvalues in the PREAR leads to an AR(2), but the mean-reversion with the smaller values of $\lambda_1 = \lambda_2 = 0.58$ is much faster than with the larger values of $\lambda_1 = \lambda_2 = 0.80$ in the CREAR example of repeated eigenvalues.

For $P = 5$, the eigenvalues associated with the OAR estimate are all less than 1 in magnitude. Hence, none of the inequality constraints for models 12 to 14 (which are as for models 2 to 14, i.e. $|\lambda_k| < 2$, local-to-unity, and non-explosive) bind in these examples; all have the same estimated eigenvalues and hence coefficients as the OAR.

In model 15, the constraint of $|\lambda_k| < 0.90$ results in a complex conjugate pair as the largest eigenvalues (and their magnitude is essentially equal to the constraint). Model 16 again provides an example of an oscillating unit root, with a wavelength of 3.75 quarters. The repeated eigenvalue example in model 17 produces a higher rate of mean reversion than the OAR(5), and the remaining eigenvalues are very similar to the complex conjugate pair (λ_2, λ_3) and λ_4 in the OAR(5)/unconstrained EAR.

The PREAR results in models 18 and 19 again show that a constraint of real eigenvalues leads to AR(1) models, with essentially the same coefficient as the $P = 4$ results in models 8 and 9. Similarly, the repeated eigenvalue constraint in model 20 essentially replicates model 10.

[19]The wavelength is obtained from the angle of the polar form for λ_1, i.e. $\cos^{-1}[\mathrm{Re}(\lambda_1)] = \cos^{-1}[-0.3213] = 1.90$ radians, hence giving the wavelength as $2\pi/1.90 = 3.31$ (and a frequency of $4/3.3 = 1.21$ cycles per year).

[20]Imposing a zero restriction via the EAR is trivial, because it simply defaults to reducing the AR(P) to and AR($P - 1$). But zero restrictions offer a promising avenue for parameter reduction in multivariate applications.

6.2 AR(P) component decomposition

Figure 4 contains the decomposition of historical data and forecasts into the eigenvalue components obtained from model 11, i.e. the OAR with $P = 5$ (which is also replicated by the EAR estimations in models 12-14). The forecast function is:

$$\mathbb{E}_t\left[y_{t+h}\right] = \lambda_1^h X_{1,t} + \left[\begin{array}{cc} 1 & 0 \end{array}\right]\left[\begin{array}{cc} \phi_2^* & \phi_3^* \\ 1 & 0 \end{array}\right]^h \left[\begin{array}{c} X_{2,t}^* \\ X_{2,t-1}^* \end{array}\right] + \lambda_4^h X_{5,t} + \lambda_4^h X_{5,t} \quad (60)$$

where t is Mar-1981, component 1 is an AR(1) with $\lambda_1 = 0.98$ and $X_{1,t} = 8.45$, component 2 is an AR(2) with $(\phi_2^*, \phi_3^*) = (-0.21, -0.91)$ and $\left(X_{2,t}^*, X_{2,t-1}^*\right) = (1.07, 3.88)$,[21] component 4 is an AR(1) with $\lambda_4 = -0.73$ and $X_{4,t} = 0.27$, and component 5 is an AR(1) with $\lambda_5 = 0.71$ and $X_{5,t} = -0.61$. The historical components are the model-inferred time series for $X_{1,t}$, $X_{2,t}^*$ (and $X_{2,t-1}^*$), $X_{4,t}$, and $X_{5,t}$ over the time range of Jun–1948 to Mar-1981. As discussed in section 5.3, these time series are the decomposed data underlying the AR models used to produce each of the forecast components. Also included for the AR(1) model for the λ_1 forecast component is the $\pm$ standard deviation obtained from the closed-form FEV expression in section 5.4.[22]

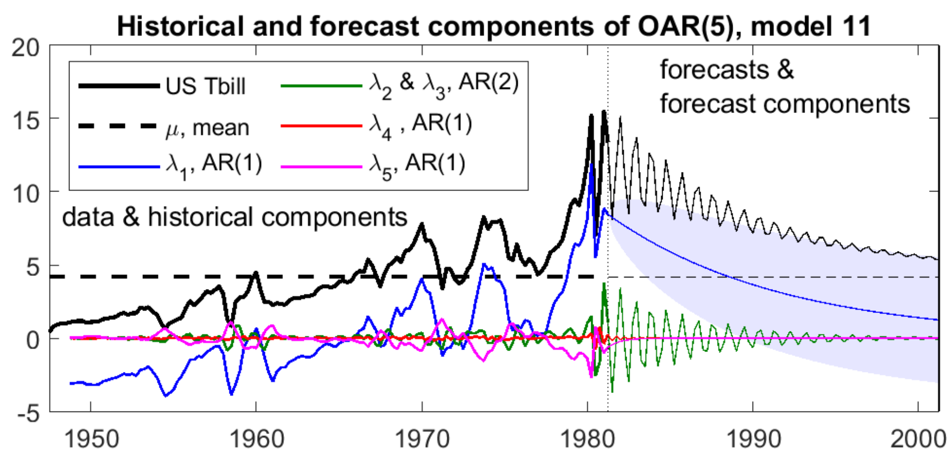

Figure 4: Decompositions of the historical data and the forecasts/IRFs for the AR(5). The historical data y_t and its components is from Jun-1947 to Mar-1981. The forecasts/IRFs $\mathbb{E}_t\left[y_{t+h}\right]$ and components are from Jun-1981. The shaded area is ± 1 standard deviation around the forecast component 1.

Regarding forecast dynamics, the AR(1) component associated with λ_1 unambiguously makes the largest contribution. That is, as shown in table 1 underneath the OAR(5) eigenvalue estimates, the λ_1 component has the largest magnitude $X_{1,t}$ and also the largest magnitude eigenvalue. The AR(2) component associated with the eigenvalue pair (λ_2, λ_3) makes the second-largest contribution, with a large pair of starting values $\left(X_{2,t}^*, X_{2,t-1}^*\right)$ at the end of the estimated sample period and the second-largest eigenvalue magnitude. Such

[21] The eigenvalue pair underlying the AR(2) component coefficients is $(\lambda_2, \lambda_3) = -0.10 \pm 0.95i$ (as shown for the OAR(5) model 11 in table 1), which has a magnitude of 0.95. $(X_{2,t}, X_{3,t}) = 0.54 \pm 1.92i$, which has a magnitude of 1.99.

[22] Because the expression $X_t = \Lambda^{P-1} V^{-1} Y_t$ uses matrices Λ^{P-1} and V^{-1} with estimated parameters, confidence intervals could be calculated for $X_{k,t}$ or $X_{k,t}$ and $X_{k,t}^*$. However, the overall summation is not subject to confidence intervals because it returns the observable variable y_t. That is, using $\mathbb{E}_t\left[y_{t+h}\right] = JV\Lambda^h V^{-1} Y_t$ from the proof of Proposition 4, $\mathbb{E}_t\left[y_{t+0}\right] = JV\Lambda^0 V^{-1} Y_t = JVV^{-1} Y_t = JY_t = y_t$.

pronounced oscillations are implausible in a forecast of interest rates, so a forecaster would likely use judgement to ignore these pronounced oscillations (which could be achieved by setting $X^*_{2,t}$ and $X^*_{2,t-1}$ to zero, or estimating a model with real eigenvalues).[23] The AR(1) components associated with λ_4 and λ_5 respectively make the third- and fourth largest forecast contributions.

Regarding overall dynamics, the ergodic variances in table 1 below the OAR(5) eigenvalue estimates show that the λ_1 component dominates, with a value of 18.88. The ergodic variance associated with (λ_2, λ_3) is the second largest. The ergodic variance for λ_5 is much larger than for λ_4, showing that the component contributions to overall dynamics do not strictly correspond to eigenvalue magnitudes. Note that the total ergodic variance $\Omega_Y(\infty)$ of the OAR(5) is 17.95, which is obviously smaller than the sum of the component ergodic variances. The difference is due to the covariances in $\Omega_Y(\infty)$, which are not shown in the table. In particular, there is a notable negative covariance term, i.e. -1.28, between the components associated with λ_1 and λ_5.

6.3 TVEAR estimation

As an example of estimating the TVOAR and TVEAR outlined in section 4.4, I set $P = 4$, calibrate the parameters $\kappa = 0.01$ and $\Omega_\varepsilon = 1$, and I use the arbitrary initialization of $\phi_0 = [0.88, -0.54, 0.84, -0.57]$ or the equivalent $x_0 = [2, 0.67, -0.67, 2]$, and $P_0 = 5I_P$. Note that x_0 simply uses linear spacing between 2 and -2, and ϕ_0 is the coefficient vector obtained from using x_0 in the CREAR expressions from section 4.3.2. The TVOAR and TVEAR therefore start from the same set of AR coefficients (the associated eigenvalues are $0.76 \pm 0.32i$ and $-0.32 \pm 0.86i$), and the other calibrations and initializations are set to be identical in the TVOAR and TVEAR so that the differences in their results are attributable to how the state variables are used to obtain the coefficients. The parameters κ and/or Ω_ε could, of course, be estimated by maximizing the log-likelihood function associated with the Kalman filter, and formal diffuse priors could be used instead of the starting covariance of $P_0 = 5I_P$, but both aspects are beyond what is required for the empirical illustration here.

The top panel of figure 5 plots the estimates of ϕ_t from the TVOAR over the sample period Jun-1947 to Sep-2008, and the bottom panel plots the absolute values of the associated eigenvalues. The latter shows that the estimated AR(4) has regular occurrences of explosive behavior, as evidenced by the periods when the eigenvalue magnitudes are above 1 (e.g. 1978 to 1980). Note that complex conjugate eigenvalues have the same magnitude, in which case pairs of eigenvalue magnitudes overlap precisely, e.g. (λ_1, λ_2) from 1990.

The top panel of figure 6 plots the estimates of x_t from the TVEAR over the sample period, the middle panel plots the associated magnitudes of the eigenvalues $\lambda[x_t, \gamma = 1]$, and the last panel plots the associated coefficients $\phi(\lambda[x_t, \gamma = 1])$. The middle panel shows that all estimated eigenvalues have $|\lambda_k| < 1$, consistent with the setting of $\gamma = 1$,

[23] The AR(2) component in this example is heavily influenced by the large fluctuations of the US Tbill rate from Sep-1979, which is associated with the Volker-led monetary policy tightening in October 1979 followed by subsequent market reactions and Federal Reserve policy responses to ongoing money growth, inflation, and output growth outcomes. More generally, the plausibility of any oscillatory component in macroeconomic forecasts could be questioned, or at least those with high frequencies, but I do not pursue that consideration in this article.

and so the associated AR(4) is therefore always non-explosive.

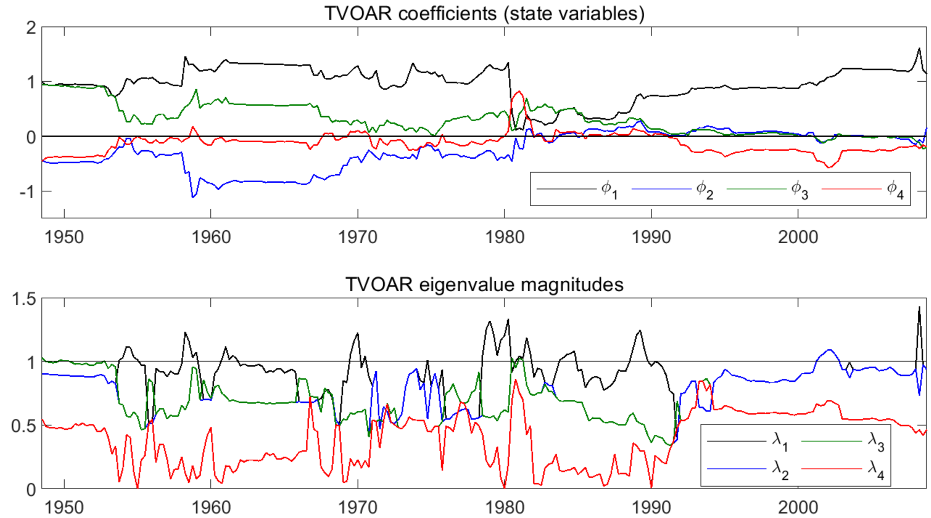

Figure 5: The results from estimating the TVOAR specified in section 4.4. The top panel plots the time series of estimated TVOAR coefficients, and the bottom panel plots the magnitudes of the eigenvalues associated with the estimated coefficients.

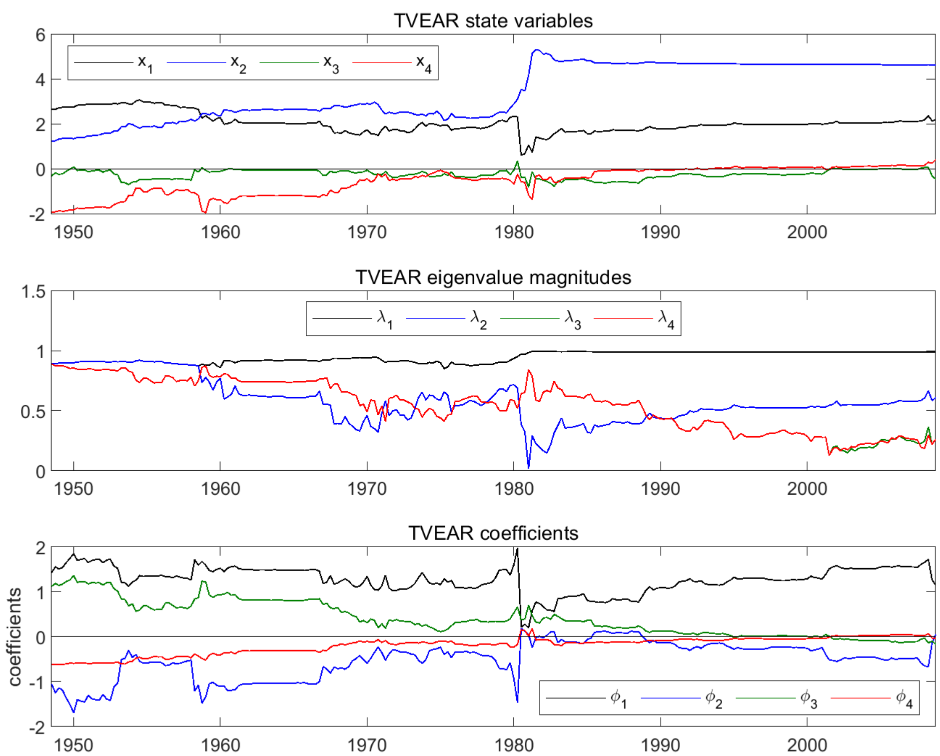

Figure 6: The results from estimating the TVEAR specified in section 4.4. The top panel plots the time series of estimated TVEAR state variables, and the middle and bottom panels respectively plot the eigenvalue magnitudes and coefficients associated with the estimated state variables.

7 Conclusion

The eigensystem autoregression (EAR) framework introduced in this article allows AR models with P lags, i.e. an AR(P), to be specified and estimated directly in terms of P eigenvalues. As such, rather than accepting the dynamics delivered by an unconstrained AR estimated via OLS (an OAR), one can specify constraints within the EAR framework that restrict the allowable dynamics of the estimated AR(P) as may be required and/or desired for the task at hand. Examples shown in the empirical application to US Treasury bill rate data are using the EAR framework to restrict an explosive OAR to be mildly explosive, non-explosive, or stationary/mean-reverting by respectively using eigenvalue magnitude constraints of local-to-unity, unity, and less than unity. Additionally, the example of applying the EAR framework with a unity magnitude constraint to a time-varying AR estimation shows that a mean-reverting dynamics may be guaranteed at all times, whereas a direct estimation with time-varying coefficients often results in periods where the dynamics are explosive.

The EAR framework also produces closed-form forecasts and impulse response functions (IRFs). This applies for any AR model, including those from an OAR once its eigenvalues are obtained. The closed-form expressions turn out to be sums of components that are themselves AR(1) or AR(2) models, and so AR(P) forecasts/IRFs and the historical data itself may be decomposed into those components. Such decompositions provide a diagnostic on the sources and contributions of the dynamics underlying an AR(P), such as the empirical example in section 6.2 that quantifies an implausible pronounced high-frequency oscillation from an AR(2) component in an AR(5) model.

Other potential applications of the EAR framework to univariate applications, in brief, include: (1) correcting downward bias in the mean-reversion rate of AR models via their eigenvalues (e.g. using median-unbiased eigenvalue estimates analogous to Andrews (1993) and Andrews and Chen (1994) for median-unbiased AR coefficients); (2) estimating seasonal factors directly within the AR model (using complex conjugate eigenvalue pairs with their magnitudes constrained to 1, as in models 6 and 16 from table 1); (3) avoiding parameter redundancy in autoregressive moving-average models (by constraining the AR and moving-average lag factors to ensure factor cancellations cannot occur);[24] (4) non-explosive bootstrapping of ARs (by estimating AR models with an eigenvalue magnitude constraint of 1 from bootstrapped samples).

The more important extension of the EAR framework is to the multivariate context, i.e. considering a vector autoregression (VAR) from the perspective of its eigensystem, which I hereafter refer to the eigensystem VAR (EVAR). Many aspects of the EAR framework carry over to the EVAR framework, such as controlling the allowable dynamics of the resulting EVAR by constraining the estimated eigenvalues, producing closed-form forecasts/IRFs, and decomposing EVAR forecasts/IRFs and historical data into components associated with AR(1) and AR(2) processes with respect to the EVAR eigenvalues. As such, the EAR applications in this article and the potential EAR applications noted above also apply to the EVAR. Additionally, the perspective provided by the estimated eigenvectors for the EVAR provides insight on how variables move together, and constraining the eigenvectors allows control over allowable variable co-movements. Both of these aspects should prove useful for VAR identification and structural modelling.

However, specifying and estimating the EVAR eigenvector parameters in conjunction

[24] See, for example, Lütkepohl (2006) section 12 for further discussion on this issue and its implications.

with the eigenvalues is more involved than for an EAR, which only requires eigenvalue estimation to obtain the AR(P) coefficients. In particular, the computationally efficient methods of vector convolution used within the EAR framework, and the even more efficient hybrid method from section 4.3.4, are no longer generally applicable in the EVAR framework; a fundamentally different estimation procedure is required. This difference along with the full discussion of eigenvectors within the EVAR are the reasons I have developed the EAR separately, i.e. an EAR is best accommodated within its own framework rather than as a single variable case within the EVAR framework.

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A Additional material and proofs for section 2

Sections A.1 to A.3 provide further details for the AR(2), respectively its companion form and associated eigensystem decompositions for the distinct and repeated eigenvalues cases, AR(2) forecast/IRF functions, and forecast/IRF functions when the eigenvalues are complex conjugate pairs. Sections A.4 and A.5 respectively provide the proofs for Propositions 1 and 2.

A.1 AR(2) companion form

The companion form for an AR(2) is:

$$\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix} \tag{61}$$

When the eigenvalues are distinct, the AR(2) companion matrix may be expressed as the eigensystem decomposition:

$$\begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \tag{62}$$

which may be verified by direct evaluation. That is, abbreviating the previous equation to $\Phi = V\Lambda V^{-1}$, the inverse of the eigenvector matrix V may be expressed as its determinate multiplied by the adjugate of V, i.e.:

$$V = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \tag{63}$$

Multiplying out $V\Lambda\text{adj}(V)$ gives:

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 - \lambda_2^2 & \lambda_1\lambda_2^2 - \lambda_1^2\lambda_2 \\ \lambda_1 - \lambda_2 & 0 \end{bmatrix} \tag{64}$$

and then multiplying the factored result of $V\Lambda\text{adj}(V)$ by the determinant gives:

$$\frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} (\lambda_1-\lambda_2)(\lambda_1+\lambda_2) & -\lambda_1\lambda_2(\lambda_1-\lambda_2) \\ \lambda_1-\lambda_2 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1+\lambda_2 & -\lambda_1\lambda_2 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \quad (65)$$

where the final result uses $(\phi_1, \phi_2) = (\lambda_1+\lambda_2, -\lambda_1\lambda_2)$ from equation 7.

When the eigenvalues are repeated, the AR(2) companion matrix may be expressed as the eigensystem decomposition:[25]

$$\begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

which may be verified by direct evaluation. That is:

$$\begin{aligned}
\begin{bmatrix} \lambda_1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} &= \begin{bmatrix} \lambda_1^2 & 2\lambda_1 \\ \lambda_1 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \\
&= \begin{bmatrix} 2\lambda_1 & -\lambda_1^2 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \qquad (66)
\end{aligned}$$

A.2 AR(2) forecasts/IRFs

The point forecasts/IRFs for an AR(2) are obtained as:

$$\mathbb{E}_t[y_{t+h}] = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}^h \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \quad (67)$$

When the eigenvalues are distinct, equation 67 may be expressed as:

$$\begin{aligned}
\mathbb{E}_t[y_{t+h}] &= \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1}\right)^h \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^h\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1^h & 0 \\ 0 & \lambda_2^h \end{bmatrix}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1^{h-1} & 0 \\ 0 & \lambda_2^{h-1} \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1^h & \lambda_2^h \end{bmatrix}\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} \\
&= \lambda_1^h X_{1,t} + \lambda_2^h X_{1,t} \qquad (68)
\end{aligned}$$

[25]See Wilkinson (1965) pp. 14-15. Hamilton (1994) pp. 18-19 specifies the.form of the eigenvalue matrix Λ, but not the eigenvector matrix V. Note that V is a confluent Vandermonde matrix and, as in footnote 13 from section 5.1 of this article, such matrices are applied in areas such as polynomial interpolation, signal processing, and control theory.

where:

$$\begin{aligned}
\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \\
&= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \\
&= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_t - \lambda_2 y_{t-1} \\ -y_t + \lambda_1 y_{t-1} \end{bmatrix} \\
&= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 y_t - \lambda_1 \lambda_2 y_{t-1} \\ -\lambda_2 y_t + \lambda_1 \lambda_2 y_{t-1} \end{bmatrix} \qquad (69)
\end{aligned}$$

The forecast/IRF expression $\lambda_1^h X_{1,t} + \lambda_2^h X_{1,t}$ in the case of distinct eigenvalues applies to pairs of real or complex conjugate eigenvalues. In the real case, $[X_{1,t}, X_{2,t}]'$ will obviously have real values, and so the AR(2) forecasts/IRFs are the sum of two AR(1) processes. The complex conjugate case produces a sum of two complex AR(1) processes which, as shown section A.3, may be re-expressed in terms of real-valued trigonometric functions.

When the eigenvalues are repeated, and so both must be real, equation 67 may be expressed as:

$$\begin{aligned}
\mathbb{E}_t [y_{t+h}] &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} \lambda_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \right)^h \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}^h \begin{bmatrix} \lambda_1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \qquad (70) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1^h & h\lambda_1^{h-1} \\ 0 & \lambda_1^h \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}^{-1} \\
&\quad \times \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1^h & h\lambda_1^{h-1} \end{bmatrix} \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} \qquad (71)
\end{aligned}$$

where:

$$\begin{aligned}
\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} &= \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \\
&= \begin{bmatrix} y_t \\ \lambda_1 y_t - \lambda_1^2 y_{t-1} \end{bmatrix} \qquad (72)
\end{aligned}$$

so $[X_{1,t}, X_{2,t}]'$ are also obviously real.

A.3 AR(2) with complex conjugate eigenvalues

When the AR(2) eigenvalues are a complex conjugate pair, $[X_{1,t}, X_{2,t}]'$ will also be a complex conjugate pair, which may be shown by direct evaluation. That is, in the expression for $[X_{1,t}, X_{2,t}]'$ for the distinct eigenvalue case, setting $\lambda_2 = \overline{\lambda_1}$ where $\lambda_1 =$

$\mathrm{Re}(\lambda_1) + i\,\mathrm{Im}(\lambda_1)$ gives:

$$\begin{aligned}
\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} &= \frac{1}{\lambda_1 - \overline{\lambda_1}} \begin{bmatrix} \lambda_1 y_t - \lambda_1 \overline{\lambda_1} y_{t-1} \\ -\overline{\lambda_1} y_t + \lambda_1 \overline{\lambda_1} y_{t-1} \end{bmatrix} \\
&= \frac{1}{2i\,\mathrm{Im}(\lambda_1)} \begin{bmatrix} [\mathrm{Re}(\lambda_1) + i\,\mathrm{Im}(\lambda_1)]\, y_t - |\lambda_1|^2 y_{t-1} \\ -[\mathrm{Re}(\lambda_1) - i\,\mathrm{Im}(\lambda_1)]\, y_t + |\lambda_1|^2 y_{t-1} \end{bmatrix} \\
&= \frac{-i}{2\,\mathrm{Im}(\lambda_1)} \begin{bmatrix} i\,\mathrm{Im}(\lambda_1)\, y_t + \left[\mathrm{Re}(\lambda_1)\, y_t - |\lambda_1|^2 y_{t-1}\right] \\ i\,\mathrm{Im}(\lambda_1)\, y_t - \left[\mathrm{Re}(\lambda_1)\, y_t - |\lambda_1|^2 y_{t-1}\right] \end{bmatrix} \\
&= \frac{1}{2\,\mathrm{Im}(\lambda_1)} \begin{bmatrix} \mathrm{Im}(\lambda_1)\, y_t - i\left[\mathrm{Re}(\lambda_1)\, y_t - |\lambda_1|^2 y_{t-1}\right] \\ \mathrm{Im}(\lambda_1)\, y_t + i\left[\mathrm{Re}(\lambda_1)\, y_t - |\lambda_1|^2 y_{t-1}\right] \end{bmatrix} \qquad (73)
\end{aligned}$$

Therefore:

$$\mathbb{E}_t[y_{t+h}] = \begin{bmatrix} \lambda_k^h & \overline{\lambda_k^h} \end{bmatrix} \begin{bmatrix} X_{1,t} \\ \overline{X_{1,t}} \end{bmatrix} \qquad (74)$$

The forecasts/IRFs for an AR(2) with complex conjugate eigenvalues may also be expressed in trigonometric form, which is a perspective that clearly shows the oscillatory nature of the contributions to forecasts/IRFs from AR(2) components. Hence, setting $(\lambda_1, \lambda_1) = r\exp(\pm i\theta)$ gives:

$$\begin{aligned}
\mathbb{E}_t[y_{t+h}] &= \begin{bmatrix} \lambda_k^h & \overline{\lambda_k^h} \end{bmatrix} \begin{bmatrix} X_{1,t} \\ \overline{X_{1,t}} \end{bmatrix} \\
&= \begin{bmatrix} [r\exp(i\theta)]^h & [r\exp(-i\theta)]^h \end{bmatrix} \begin{bmatrix} \mathrm{Re}(X_{1,t}) + i\,\mathrm{Im}(X_{1,t}) \\ \mathrm{Re}(X_{1,t}) - i\,\mathrm{Im}(X_{1,t}) \end{bmatrix} \\
&= r^h \begin{bmatrix} \exp(ih\theta) & \exp(-ih\theta) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(X_{1,t}) + i\,\mathrm{Im}(X_{1,t}) \\ \mathrm{Re}(X_{1,t}) - i\,\mathrm{Im}(X_{1,t}) \end{bmatrix} \qquad (75)
\end{aligned}$$

and $\exp(ih\theta)$ may be converted to trigonometric using the Euler formula, i.e.:

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta) \qquad (76)$$

and the De Moivre formula, i.e.:

$$[\cos(\theta) + i\sin(\theta)]^h = \cos(h\theta) + i\sin(h\theta) \qquad (77)$$

That is:

$$\begin{aligned}
\exp(ih\theta) &= [\exp(i\theta)]^h \\
&= [\cos(\theta) + i\sin(\theta)]^h \\
&= \cos(h\theta) + i\sin(h\theta) \qquad (78)
\end{aligned}$$

Setting $\mathrm{Re}(X_{1,t}) + i\,\mathrm{Im}(X_{1,t}) = a + ib$ for notational convenience:

$$\begin{aligned}
& r^h \begin{bmatrix} \exp(ih\theta) & \exp(-ih\theta) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(X_{1,t}) + i\,\mathrm{Im}(X_{1,t}) \\ \mathrm{Re}(X_{1,t}) - i\,\mathrm{Im}(X_{1,t}) \end{bmatrix} \\
&= r^h \begin{bmatrix} \cos(h\theta) + i\sin(h\theta) & \cos(h\theta) - i\sin(h\theta) \end{bmatrix} \begin{bmatrix} a + ib \\ a - ib \end{bmatrix} \\
&= r^h [a\cos(h\theta) + ib\cos(h\theta) + ia\sin(h\theta) - b\sin(h\theta) \\
&\quad + a\cos(h\theta) - ib\cos(h\theta) - ia\sin(h\theta) - b\sin(h\theta)] \\
&= r^h [2a\cos(h\theta) - 2b\sin(h\theta)] \qquad (79)
\end{aligned}$$

and therefore:

$$\lambda_1^h X_{1,t} + \lambda_2^h X_{2,t} = 2r^h \left[\text{Re}\left(X_{1,t}\right)\cos\left(h\theta\right) - \text{Im}\left(X_{1,t}\right)\sin\left(h\theta\right)\right] \tag{80}$$

This result matches that obtained in Hamilton (1994) pp. 14-16.

A.4 Generalized AR(2) triangle

Proof of Proposition 1. Given the expression for the eigenvalues of the AR(2) from equation 6 of section 2.2, the objective is to constrain their magnitudes to a maximum of γ, i.e.:

$$\left(\left|\lambda_1\right|, \left|\lambda_2\right|\right) = \left|\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}\right| < \gamma \tag{81}$$

The eigenvalues can either be a distinct real pair if $\phi_1^2 + 4\phi_2 > 0$, a repeated real pair if $\phi_1^2 + 4\phi_2 = 0$, or a complex conjugate pair if $\phi_1^2 + 4\phi_2 < 0$. For the distinct real eigenvalue case, the largest eigenvalue will be $\frac{1}{2}\left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)$, which needs to be less than γ, hence:

$$\begin{aligned} \phi_1 + \sqrt{\phi_1^2 + 4\phi_2} &< 2\gamma \\ \sqrt{\phi_1^2 + 4\phi_2} &< 2\gamma - \phi_1 \\ \phi_1^2 + 4\phi_2 &< \left(2\gamma - \phi_1\right)^2 \\ \phi_1^2 + 4\phi_2 &< 4\gamma^2 - 4\gamma\phi_1 + \phi_1^2 \\ \phi_2 &< \gamma^2 - \gamma\phi_1 \end{aligned} \tag{82}$$

and the smallest eigenvalue will be $\frac{1}{2}\left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right)$, which needs to be greater than $-\gamma$, hence:

$$\begin{aligned} \phi_1 - \sqrt{\phi_1^2 + 4\phi_2} &> -2\gamma \\ -\sqrt{\phi_1^2 + 4\phi_2} &> -2\gamma - \phi_1 \\ \phi_1^2 + 4\phi_2 &< \left(2\gamma + \phi_1\right)^2 \\ \phi_1^2 + 4\phi_2 &< 4\gamma^2 + 4\gamma\phi_1 + \phi_1^2 \\ \phi_2 &< \gamma^2 + \gamma\phi_1 \end{aligned} \tag{83}$$

In summary, $\phi_2 < \gamma^2 - \gamma\phi_1$ and $\phi_2 > \gamma^2 - \gamma\phi_1$, so $\phi_2 < \gamma^2 - \gamma\left|\phi_1\right| = \gamma\left(\gamma - \left|\phi_1\right|\right)$.

For the repeated real eigenvalue case, both eigenvalues will be $\frac{1}{2}\phi_1$, which needs to have an absolute value less than γ. Hence:

$$-2\gamma < \phi_1 < 2\gamma \tag{84}$$

For the complex eigenvalue case, the modulus of both eigenvalues is calculated as $|a \pm ib| = \sqrt{a^2 + b^2}$, where $a = \frac{1}{2}\phi_1$ and $b = -\left(\phi_1^2 + 4\phi_2\right)$. Hence, setting the modulus to be less than γ gives:

$$\begin{aligned}\sqrt{\frac{\phi_1^2}{4}-\frac{\phi_1^2+4\phi_2}{4}} &< \gamma \\ \sqrt{-\phi_2} &< \gamma \\ -\phi_2 &< \gamma^2 \\ \phi_2 &> -\gamma^2 \end{aligned} \tag{85}$$

■

A.5 Real eigenvalue region of AR(2) stability triangle

Proof of Proposition 2. From equation 7 of 2.2, i.e. $(\phi_1,\phi_2)=(\lambda_1+\lambda_2,-\lambda_1\lambda_2)$, two positive eigenvalues will result in $(\phi_1>0,\phi_2<0)$, which is the bottom-right quadrant of the (ϕ_1,ϕ_2) plot in figure 2. The region is then defined by the complex and right-hand boundaries of the stability triangle, i.e. respectively $\phi_2=-\frac{1}{4}\phi_1^2$ and $1-\phi_1$.

The case for two negative eigenvalues is analogous to the case for two positive eigenvalues, i.e. equation 7 will result in $(\phi_1<0,\phi_2<0)$, which is the bottom-left quadrant of the (ϕ_1,ϕ_2) plot. The region is then defined by the complex boundary and left-hand boundary $1+\phi_1$ of the stability triangle.

The case for one positive and one negative eigenvalue results in $\phi_2>0$, which is in the top two quadrants of the (ϕ_1,ϕ_2) plot. The region is defined by the left- and right-hand boundaries of the stability triangle. ■

B Additional material and proofs for section 4

This appendix provides details for the CMLE of the OAR and EAR models in section 4, and also the proof of Proposition 3. Section B.1 provides the exposition that leads to an OAR, and section B.2 provides the exposition that leads to the NLS estimates of an EAR (i.e. a PREAR or CREAR). Sections B.3 and B.4 respectively show that the Jacobians for the NLS estimation of the PREAR and CREAR are amenable to analytic calculation, should additional computational efficiency be required in any application. For notational convenience and clarity in sections B.2 to B.5, I abbreviate $\log(\mathcal{L}[\Theta,\Omega])$ to $\log(\mathcal{L})$, and $\phi(\lambda[x])$ to $\phi(\lambda)$ or $\phi(x)$ depending on the context.

B.1 CMLE to produce OAR parameter estimates

For the OAR log-likelihood function, the substitution of $\varepsilon_t=y_t-\phi Y_{t-1}$ from equation 12 into equation 15 gives:

$$\begin{aligned}\log(\mathcal{L}) &= -\frac{T}{2}\log(2\pi)-\frac{T}{2}\log(\Omega)-\frac{1}{2\Omega}\sum_{t=1}^{T}\left[y_t-\phi Y_{t-1}\right]^2 \\ &= -\frac{T}{2}\log(2\pi)-\frac{T}{2}\log(\Omega)-\frac{1}{2\Omega}\left[y-\phi Y_{-1}\right]\left[y-\phi Y_{-1}\right]' \\ &= -\frac{T}{2}\log(2\pi)-\frac{T}{2}\log(\Omega)-\frac{1}{2\Omega}\left[yy'-2yY_{-1}'\phi'+\phi Y_{-1}Y_{-1}'\phi'\right]\end{aligned} \tag{86}$$

where the second line expresses the summation as the inner product of the vector $y - \phi Y_{-1}$, with y being a $1 \times T$ vector containing all y_t, and Y_{-1} being a $P \times T$ matrix containing all Y_{t-1}. Note also that the third line uses $yY'_{-1}\phi' = \phi Y_{-1}y'$ (given both expressions are scalars, so their transposes are equal to each other) to obtain $-yY'_{-1}\phi' - \phi Y_{-1}y' = -2yY'_{-1}\phi'$.

To find the values of ϕ that maximize $\log(\mathcal{L})$, differentiate $\log(\mathcal{L})$ with respect to ϕ' and set the result to zero, i.e.:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \phi'} \log(\mathcal{L}) \\ &= -0 - 0 - \frac{\partial}{\partial \phi'}\left(\frac{1}{2\Omega}\left[yy' - 2yY'_{-1}\phi' + \phi Y_{-1}Y'_{-1}\phi'\right]\right) \\ &= 2yY'_{-1} + 2\phi Y_{-1}Y'_{-1} \\ \phi Y_{-1}Y'_{-1} &= yY'_{-1} \\ \phi &= yY'_{-1}\left(Y_{-1}Y'_{-1}\right)^{-1} \end{aligned} \tag{87}$$

Note that for the third line, I have used the following matrix calculus results:[26]

$$\frac{\partial (a'u)}{\partial u} = a' \tag{88a}$$

$$\frac{\partial (u'Au)}{\partial u} = 2u'A \tag{88b}$$

where a and u are generic column vectors, and A is a generic square matrix.

With $(y - \phi Y_{-1})(y - \phi Y_{-1})' = \varepsilon\varepsilon'$, the value of Ω that maximizes $\log(L)$ is calculated as:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \Omega}\log(L) \\ &= -\frac{T}{2}\frac{\partial \log(\Omega)}{\partial \Omega} - \frac{1}{2}\frac{\partial \Omega^{-1}}{\partial \Omega}\varepsilon\varepsilon' \\ &= -T\Omega^{-1} + \Omega^{-2}\varepsilon\varepsilon' \\ T\Omega &= \varepsilon\varepsilon' \\ \Omega &= \frac{1}{T}\varepsilon\varepsilon' \end{aligned} \tag{89}$$

Note that the expressions in equations 87 and 89 are transposes of the typical OLS forms that apply when data are expressed in columns. I have specified the data for the AR to be in rows, which is consistent with the format for vector autoregressions.

B.2 CMLE to produce PREAR parameter estimates

For the PREAR optimization, substituting $\varepsilon_t = y_t - \phi(x) Y_{t-1}$ from equation 21 in section 4.3.1, or the analogous expression for the CREAR in section 4.3.2, into equation 15 and re-arranging as in section B.1 gives the following log-likelihood function:

$$\begin{aligned} \log(\mathcal{L}) &= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\Omega) \\ &\quad -\frac{1}{2\Omega}\left[y - \phi(x) Y_{-1}\right]\left[y - \phi(x) Y_{-1}\right]' \end{aligned} \tag{90}$$

[26] See Petersen and Pedersen (2012), for example, but note that I use the numerator layout convention. Also note that the vector ϕ is in row form, which is why I have taken the differential with respect to ϕ'.

In this case, $\log(\mathcal{L})$ is a non-linear function of x, so the values of x that maximize $\log(\mathcal{L})$ cannot be obtained analytically. However, as is typically done, an initial value of x may be improved upon iteratively until a specified condition for convergence. The linear approximation for x is obtained from a first-order Taylor expansion for $\phi(x)$ around starting value of x_0, i.e.:

$$\begin{aligned}\phi(x) &\simeq \phi(x_0) + [x - x_0] \left.\frac{\partial[\phi(x)]'}{\partial x'}\right|_{x=x_0} \\ &= \phi(x_0) - x_0\frac{\partial[\phi(x_0)]'}{\partial x'} + x\frac{\partial[\phi(x_0)]'}{\partial x'} \qquad (91)\end{aligned}$$

where the expression $\partial[\phi(x)]'/\partial x'$ is the Jacobian of $[\phi(x)]'$ with respect to x', which is the following $P \times P$ matrix:

$$\frac{\partial[\phi(x)]'}{\partial x'} = \begin{bmatrix} \frac{\partial[\phi(x)]_1}{\partial x_1} & \cdots & \frac{\partial[\phi(x)]_1}{\partial x_P} \\ \vdots & \cdots & \vdots \\ \frac{\partial[\phi(x)]_P}{\partial x_1} & \cdots & \frac{\partial[\phi(x)]_P}{\partial x_P} \end{bmatrix} \qquad (92)$$

and $\partial[\phi(x_0)]'/\partial x'$ is $\partial[\phi(x)]'/\partial x'$ evaluated at $x = x_0$.

Substituting the approximation for $\phi(x)$ into $y - \phi(x)Y_{-1}$ gives:

$$\begin{aligned}y - \phi(x)Y_{-1} &\simeq y - \left(\phi(x_0) - x_0\frac{\partial[\phi(x_0)]'}{\partial x'} + x\frac{\partial[\phi(x_0)]'}{\partial x'}\right)Y_{-1} \\ &= y - \left(\phi(x_0) - x_0\frac{\partial[\phi(x_0)]'}{\partial x'}\right)Y_{-1} - x\frac{\partial[\phi(x_0)]'}{\partial x'}Y_{-1} \\ &= y^* - xY^*_{-1} \qquad (93)\end{aligned}$$

where:

$$y^* = y - \left(\phi(x_0) - x_0\frac{\partial[\phi(x_0)]'}{\partial x'}\right)Y_{-1} \qquad (94a)$$

$$Y^*_{-1} = \frac{\partial[\phi(x_0)]'}{\partial x'}Y_{-1} \qquad (94b)$$

Using $y^* - xY^*_{-1}$ within $\log(\mathcal{L})$ obtains the following:

$$\log(\mathcal{L}) = -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\Omega) - \frac{1}{2\Omega}\left[y^* - xY^*_{-1}\right]\left[y^* - xY^*_{-1}\right]' \qquad (95)$$

which is linear with respect to x. Therefore, the same steps as in section B.1 can be followed to obtain estimates of x and Ω given the starting value of x_0, i.e.:

$$x = y^*Y^{*\prime}_{-1}\left(Y^*_{-1}Y^{*\prime}_{-1}\right)^{-1} \qquad (96a)$$

$$\Omega = \frac{1}{T}\varepsilon(x_0)[\varepsilon(x_0)]' \qquad (96b)$$

where $\varepsilon(x_0) = y^* - xY^*_{-1}$. The process may be iterated to convergence by setting x_0 to the estimate x and repeating the process until the specified convergence criteria have been met.

Note that the process outlined above could also be followed for an EAR with unconstrained eigenvalues, i.e. where $y_t = \phi(\lambda)Y_{t-1} + \varepsilon_t$ from equation 17. However, as discussed in section 4.3.5, the additional computational expense makes such an optimization redundant relative to OAR estimation.

B.3 PREAR analytic Jacobian

The simple functional forms used to obtain the eigenvalue set λ and the coefficient set ϕ for the PREAR means that the Jacobian required for the NLS estimation is amenable to analytic calculation. That is, the elements of the Jacobian may be obtained by using the chain rule:

$$\frac{\partial \left[\phi\left(x\right)\right]_i'}{\partial x_j} = \frac{\partial \left[\phi\left(\lambda\right)\right]_i'}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial x_j} \tag{97}$$

Regarding the partial differential $\partial \left[\phi\left(\lambda\right)\right]_i' / \partial \lambda_j$, the entire column $\partial \left[\phi\left(\lambda\right)\right]' / \partial \lambda_j$ (i.e. for all of the $\phi\left(\lambda\right)$ elements) may be obtained by differentiating the product of the eigenvalue lag factors from equation 10, i.e.:

$$\begin{aligned}\frac{\partial}{\partial \lambda_j}\prod_{k=1}^{P}\left(1-\lambda_k L\right) &= \frac{\partial}{\partial \lambda_j}\left(1-\lambda_j L\right)\prod_{k=1,k\neq j}^{P}\left(1-\lambda_k L\right)\\ &= -L\prod_{k=1,k\neq j}^{P}\left(1-\lambda_k L\right)\end{aligned} \tag{98}$$

The product from the resulting expression may be evaluated as for $\phi\left(\lambda\right)$ itself, i.e. by taking the convolution of all factors $[1, -\lambda_k]$ except for the $k = j$ factor which is replaced with $[0, -1]$. The resulting vector will again contain $P+1$ elements, with the first element now equal to zero. Negating the remaining P elements obtains $\partial \left[\phi\left(\lambda\right)\right]' / \partial \lambda_j$. In practice, it is more computationally efficient to obtain the convolution for $\partial \left[\phi\left(\lambda\right)\right]' / \partial \lambda_j$ from the original convolution results that are already available for $\phi\left(\lambda\right)$, rather than undertake the full alternative convolution. That is, apply vector deconvolution to the original $P+1$ vector $w_P = \left[1, -\phi_1\left(\lambda\right), \ldots, -\phi_P\left(\lambda\right)\right]$ and the vector $[1, -\lambda_j]$ to get the P vector w_P^* that represents $\prod_{k=1,k\neq j}^{P}\left(1-\lambda_k L\right)$.[27] It turns out that w_P^* is already the required result for $\partial \left[\phi\left(\lambda\right)\right]' / \partial \lambda_j$. That is, if w_P^* was convolved with $[0, -1]$, it would obtain the $P+1$ vector $[0, -w_P^*]$ that represents $-L\prod_{k=1,k\neq j}^{P}\left(1-\lambda_k L\right)$. Ignoring the first entry of zero from $[0, -w_P^*]$ and negating the remaining P elements returns w_P^*.

The partial differential $\partial \lambda_j / \partial x_j$, which is a scalar result, may be evaluated from the scaled logistic function as follows:

$$\begin{aligned}\lambda_j &= \gamma\left[1+\exp\left(-x_j\right)\right]^{-1}\\ \frac{\partial}{\partial x_j}\left[1+\exp\left(-x_j\right)\right]^{-1} &= \gamma\frac{\partial}{\partial u}u^{-1}\frac{\partial}{\partial x_j}\left[1+\exp\left(-x_j\right)\right]\\ &= \gamma\times -u^{-2}\times -\exp\left(-x_j\right)\\ &= \gamma\frac{\exp\left(-x_j\right)}{\left[1+\exp\left(-x_j\right)\right]^2}\end{aligned} \tag{99}$$

where the second line uses the chain rule with $u = \left[1+\exp\left(-x_j\right)\right]^{-1}$.

[27] The vector deconvolution algorithm is routine for dividing an algebraic polynomial by another polynomial. The MatLab function is "deconv(w_P, $[1, -\lambda_j]$)", otherwise the algorithm is straightforward to code as a double summation.

Combining the results for $\partial\partial\left[\phi\left(\lambda\right)\right]'/\partial\lambda_j$ (i.e. the vector w_P^*) and $\partial\lambda_j/\partial x_j$ (a scalar) gives the final result for the entire column $\partial\left[\phi\left(\lambda\right)\right]'/\partial x_j$ as:

$$\begin{aligned}\frac{\partial\left[\phi\left(x\right)\right]'}{\partial x_j} &= \frac{\partial\left[\phi\left(\lambda\right)\right]'}{\partial\lambda_j}\frac{\partial\lambda_j}{\partial x_j} \\ &= \left(w_P^*\right)'\frac{\exp\left(-x_j\right)}{\left[1+\exp\left(-x_j\right)\right]^2}\end{aligned} \tag{100}$$

B.4 CREAR analytic Jacobian

Obtaining an analytic Jacobian matrix for the CREAR is more involved than for the PREAR given that processing the pairs of (x_k, x_{k+1}) that underlie the AR(2) coefficients $\left(\phi_k^*, \phi_{k+1}^*\right)$ involves a logistic function for $\overline{\phi}_{k+1}^*$ nested in the logistic function for $\overline{\phi}_k^*$. Nevertheless, the function forms remain simple enough to obtain relatively succinct analytic results.

First, express the AR(P) as the convolution of second-order lag polynomials associated with the sets of AR(2) coefficients $\left(\phi_k^*, \phi_{k+1}^*\right)$, i.e.:

$$\phi\left(x\right) = \prod_{k=1 \text{ step } 2}^{P/2}\left(1-\phi_k^* L - \phi_{k+1}^* L^2\right) \tag{101}$$

where k is now an index for each pair of AR(2) coefficients $\left(\phi_{1,k}^*, \phi_{2,k}^*\right)$, "$k = 1$ step 2" is notation to represent each of those coefficient pairs, and $P/2$ represents the number of pairs. The case for when P is an odd number is discussed at the end of this section.

Analogous to the PREAR, the partial differential of the entire column $\partial\left[\phi\left(\lambda\right)\right]'/\partial\lambda_j$ may be calculated by differentiating the lag polynomial, i.e.:

$$\frac{\partial\phi\left(x\right)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(1-\phi_j^* L - \phi_{j+1}^* L^2\right)\prod_{k=1 \text{ step } 2,\ k\neq j}^{P/2}\left(1-\phi_k^* L - \phi_{k+1}^* L^2\right) \tag{102}$$

and the entire line of $\partial\phi\left(x\right)/\partial x_{j+1}$ is calculated similarly. Both calculations require ϕ_j^* and ϕ_{j+1}^* to be expressed in terms of x_j and x_{j+1}. For the coefficient $\phi_{1,k}^*$, equation 23 from section 4.3.2 is:

$$\phi_k^* = 2\gamma\left(\frac{2}{1+\exp\left(-x_k\right)} - 1\right) \tag{103}$$

For the coefficient $\phi_{2,k}^*$, equation 25 from section 4.3.2 is:

$$\phi_{k+1}^* = \frac{\overline{\phi}_{1,k+1}^* + \gamma^2}{1+\exp\left(-x_{k+1}\right)} - \gamma^2 \tag{104}$$

with $\overline{\phi}^*_{1,k+1} = \gamma\left(\gamma - \left|\phi^*_k\right|\right)$. Therefore:

$$\begin{aligned}
\phi^*_{k+1} &= \frac{\gamma\left(\gamma - \left|\phi^*_k\right|\right) + \gamma^2}{1 + \exp\left(-x_{k+1}\right)} - \gamma^2 \\
&= \frac{2\gamma^2 - \gamma\left|\phi^*_k\right|}{1 + \exp\left(-x_{k+1}\right)} - \gamma^2 \\
&= \frac{2\gamma^2 - \gamma\left|2\gamma\left(\frac{2}{1+\exp(-x_k)} - 1\right)\right|}{1 + \exp\left(-x_{k+1}\right)} - \gamma^2 \\
&= \frac{2\gamma^2\left(1 - \left|\frac{2}{1+\exp(-x_k)} - 1\right|\right)}{1 + \exp\left(-x_{k+1}\right)} - \gamma^2 \qquad (105)
\end{aligned}$$

In summary, each second-order lag polynomial $1 - \phi^*_k L - \phi^*_{k+1} L^2$ therefore has the following dependence on x_k and x_{k+1}:

$$\begin{aligned}
&1 - \phi^*_k L - \phi^*_{k+1} L^2 \\
= \;& 1 - 2\gamma\left(\frac{2}{1 + \exp\left(-x_k\right)} - 1\right) L - \left(\frac{2\gamma^2\left(1 - \left|\frac{2}{1+\exp(-x_k)} - 1\right|\right)}{1 + \exp\left(-x_{k+1}\right)} - \gamma^2\right) L^2 \qquad (106)
\end{aligned}$$

The partial differential of $1 - \phi^*_k L - \phi^*_{k+1} L^2$ with respect to x_k is:

$$\frac{\partial}{\partial x_k}\left(1 - \phi^*_k L - \phi^*_{k+1} L^2\right) = -\frac{\partial}{\partial x_k}\phi^*_k L - \frac{\partial}{\partial x_k}\phi^*_{k+1} L^2 \qquad (107)$$

where:

$$\begin{aligned}
\frac{\partial}{\partial x_k}\phi^*_{1,k} &= 2\gamma\frac{\partial}{\partial x_k}\left(\frac{2}{1 + \exp\left(-x_k\right)} - 1\right) \\
&= 4\gamma\frac{\exp\left(-x_k\right)}{\left[1 + \exp\left(-x_k\right)\right]^2} \qquad (108)
\end{aligned}$$

and:

$$\begin{aligned}
\frac{\partial}{\partial x_k}\phi^*_{k+1} &= \frac{\partial}{\partial x_k}\left(\frac{2\gamma^2\left(1 - \left|\frac{2}{1+\exp(-x_k)} - 1\right|\right)}{1 + \exp\left(-x_{k+1}\right)} - \gamma^2\right) \\
&= \frac{2\gamma^2}{1 + \exp\left(-x_{k+1}\right)} \cdot \frac{\partial}{\partial x_k}\left(1 - \left|\frac{2}{1 + \exp\left(-x_k\right)} - 1\right|\right) \\
&= -\frac{2\gamma^2}{1 + \exp\left(-x_{k+1}\right)} \cdot \frac{\partial}{\partial x_k}\left|\frac{2}{1 + \exp\left(-x_k\right)} - 1\right| \\
&= -\frac{2\gamma^2}{1 + \exp\left(-x_{k+1}\right)} \cdot \frac{\exp\left(-x_k\right)}{\left[1 + \exp\left(-x_k\right)\right]^2} \cdot \text{signum}\left[1 - \exp\left(-x_k\right)\right] \qquad (109)
\end{aligned}$$

where signum $(\cdot)$ is the sign function, and signum $\left[\frac{2}{1+\exp(-x_k)} - 1\right] =$ signum $\left[\frac{1-\exp(-x_k)}{1+\exp(-x_k)}\right] =$ signum $\left[1 - \exp\left(-x_k\right)\right]$ because $1 + \exp\left(-x_k\right)$ is always positive.

The lag polynomial $\partial\left(1-\phi_j^* L-\phi_{j+1}^* L^2\right)/\partial x_j$ may be represented as the vector $u=\left[0,-\partial\phi_k^*/\partial x_k,-\partial\phi_{k+1}^*/\partial x_k\right]$, with the elements provided by the results for $\partial\phi_k^*/\partial x_k$ and $\partial\phi_{k+1}^*/\partial x_k$. To obtain $\partial\left[\phi\left(x\right)\right]'/\partial x_j$, first deconvolve the original convolution vector $w_P=\left[1,-\phi_1\left(\lambda\right),\ldots,-\phi_P\left(\lambda\right)\right]$ by $\left[1,-\phi_k^*,\phi_{k+1}^*\right]$ to obtain the vector v and then convolve u with v. The result is a $P+1$ vector with the first element equal to zero, and negating the remaining P elements obtains $\partial\left[\phi\left(x\right)\right]'/\partial x_j$.

The partial differential of $1-\phi_k^* L-\phi_{k+1}^* L^2$ with respect to x_{k+1} is:

$$\frac{\partial}{\partial x_{k+1}}\left(1-\phi_k^* L-\phi_{k+1}^* L^2\right)=-\frac{\partial}{\partial x_{k+1}}\phi_{k+1}^* L^2 \tag{110}$$

where $\partial\phi_k^*/\partial x_{k+1}=0$ and:

$$\begin{aligned}\frac{\partial}{\partial x_{k+1}}\phi_{k+1}^* &= 2\gamma^2\left(1-\left|\frac{2}{1+\exp\left(-x_k\right)}-1\right|\right)\cdot\frac{\partial}{\partial x_{k+1}}\left[1+\exp\left(-x_{k+1}\right)\right]^{-1}\\ &= 2\gamma^2\left(1-\left|\frac{2}{1+\exp\left(-x_k\right)}-1\right|\right)\cdot\frac{\exp\left(-x_{k+1}\right)}{\left[1+\exp\left(-x_{k+1}\right)\right]^2}\end{aligned} \tag{111}$$

With these partial differential results, the lag polynomial $\partial\left(1-\phi_j^* L-\phi_{j+1}^* L^2\right)/\partial x_{j+1}$ is represented as the vector $\left[0,0,-\phi_{k+1}^*/\partial x_{k+1}\right]$. To obtain $\partial\left[\phi\left(x\right)\right]'/\partial x_{j+1}$, first deconvolve the original convolution vector as described earlier for $\partial\left[\phi\left(x\right)\right]'/\partial x_j$, convolve that result by $\left[0,0,-\phi_{k+1}^*/\partial x_{k+1}\right]$ to obtain the $P+1$ vector, and negate the last P elements to obtain $\partial\left[\phi\left(x\right)\right]'/\partial x_{j+1}$.

When P is an odd number, the last line will be the partial differential $\partial\phi\left(x\right)/\partial x_P$. This would be calculated as for the PREAR, i.e.:

$$\frac{\partial\left[\phi\left(x\right)\right]'}{\partial x_P}=\left(w_P^*\right)'\frac{\exp\left(-x_P\right)}{\left[1+\exp\left(-x_P\right)\right]^2} \tag{112}$$

B.5 Hybrid CREAR

Proof of Proposition 3. Begin with the AR(P) expressed as P eigenvalue factors, i.e. from equation 10, and factor this into the product of $(P-K)$ and K eigenvalue factors, i.e.:

$$\begin{aligned}\left[\prod_{k=1}^{P}\left(1-\lambda_k L\right)\right]y_t &= \varepsilon_t\\ \left[\prod_{k=K+1}^{P}\left(1-\lambda_k L\right)\right]\left[\prod_{k=1}^{K}\left(1-\lambda_k L\right)\right]y_t &= \varepsilon_t\\ \left[\prod_{k=1}^{P-K}\left(1-\lambda_k L\right)\right]\left[\prod_{k=1}^{K}\left(1-\lambda_k L\right)\right]y_t &= \varepsilon_t\end{aligned} \tag{113}$$

Expand both of the lag eigenvalue factor products into their equivalent lag polynomials (i.e. as in equation 11), and apply the lag operator from the K-order polynomial to y_t, i.e.:

$$\begin{aligned}\left[1-\sum_{p=1}^{P-K}\theta_p L^p\right]\left[1-\sum_{p=1}^{K}\delta_p L^p\right]y_t &= \varepsilon_t \\ \left[1-\sum_{p=1}^{P-K}\theta_p L^p\right]\left[y_t-\sum_{p=1}^{K}\delta_p y_{t-p}\right] &= \varepsilon_t \end{aligned} \tag{114}$$

Use δ_p and $y_t,\ldots,y_{t-p}$ to define the variable z_t as follows:

$$z_t = y_t - \sum_{p=1}^{K}\delta_p y_{t-p} \tag{115}$$

Apply the lag operator from the $(P-K)$-order polynomial to the variable z_t to obtain the OLS regression form for z_t, i.e.:

$$\begin{aligned}\left[1-\sum_{p=1}^{P-K}\theta_p L^p\right]z_t &= \varepsilon_t \\ z_t &= \left[\sum_{p=1}^{P-K}\theta_p L^p\right]z_t+\varepsilon_t \\ z_t &= \sum_{p=1}^{P-K}\theta_p z_{t-p}+\varepsilon_t \end{aligned} \tag{116}$$

where the final line is the supplementary OAR with the variable $z_t = y_t - \sum_{p=1}^{K}\delta_p y_{t-p}$. ■

C Additional material and proofs for section 5

This appendix provides details related to the closed-form forecasts and decompositions for section 5. Section C.1 first establishes a proposition that $X_{k,t}$ and $X_{k+1,t}$ will be a complex conjugate pair when λ_k^h and λ_{k+1}^h are a complex conjugate pair. While both aspects in the proposition are intuitively apparent, respectively because the product $VV^{-1}=I_P$ needs to have only have real elements and $\lambda_k^h X_{k,t}+\lambda_{k+1}^h X_{k+1,t}$ should contribute a real number to the forecasts, I'm not aware of a reference that explicitly establishes these required properties.[28] Section C.2 then provides the proof of Proposition 5, section C.3 provides the trigonometric form for AR(2) forecast/IRF components, and section C.4 provides the proof of Proposition 7.

C.1 X_t with pairs of complex conjugate eigenvalues

Proposition C.1 *The complex conjugate columns k and $k+1$ of the eigenvector matrix V are paired with complex conjugate rows k and $k+1$ in the inverse V^{-1}, and therefore*

[28] Hamilton (1994) pp. 15-16 establishes that the constants c_1 and c_2 associated with $c_1\lambda_1^h + c_2\lambda_2^h$ for an AR(2) will be a complex conjugate pair if $(\lambda_k,\lambda_{k+1})$ is a complex conjugate pair, and section A.3 of appendix A shows this by directly calculating $(X_{k,t},X_{k+1,t})$. Section C.1 establishes the more general case for an AR(2) component within an AR(P).

the $(X_{k,t}, X_{k+1,t})$ *pair associated with the complex conjugate eigenvalue pair* $(\lambda_k, \lambda_{k+1})$ *will themselves be a complex conjugate pair.*

Proof.

$$X_t = \Lambda^{P-1} V^{-1} Y_t \tag{117}$$

For notational convenience, first consider a specific case where there is only a single pair of complex conjugate eigenvalues and they are arranged to be the first two entries, i.e. $\lambda = \left[\lambda_1, \overline{\lambda_1}, \lambda_3, \ldots, \lambda_P\right]$. The eigenvalue matrix Λ is therefore:

$$\Lambda = \mathrm{diag}\left(\left[\lambda_1, \overline{\lambda_1}, \lambda_3, \ldots, \lambda_P\right]\right) \tag{118}$$

and the eigenvector matrix V will contain a complex conjugate pair of eigenvectors in its first two columns. Define a block-diagonal permutation matrix A as follows:

$$A = \mathrm{diag}\left([A_2, I_{P-2}]\right) \tag{119}$$

where I_{P-2} is the $(P-2) \times (P-2)$ identity matrix, A_2 is:

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A_2^{-1} \tag{120}$$

and $A_2 = A_2^{-1}$ is apparent from A_2^2 being the 2×2 identity matrix. The product VA interchanges the first two columns of V, which results in $\overline{V}$. Therefore, using $VA = \overline{V}$ and taking its inverse gives:

$$\begin{aligned} (VA)^{-1} &= \left(\overline{V}\right)^{-1} \\ A^{-1}V^{-1} &= \overline{V^{-1}} \\ AV^{-1} &= \overline{V^{-1}} \end{aligned} \tag{121}$$

The product AV^{-1} interchanges the first two rows of V^{-1}, and so $AV^{-1} = \overline{V^{-1}}$ establishes that the first two rows of V^{-1} are a complex conjugate pair. Denote these two rows as $[V^{-1}]_1$ and $[V^{-1}]_2 = \overline{[V^{-1}]_1}$, and then the first two rows of $X_t = \Lambda^{P-1}V^{-1}Y_t$ will be:

$$\begin{aligned} \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \overline{\lambda_1} \end{bmatrix}^{P-1} \begin{bmatrix} [V^{-1}]_1 \\ \overline{[V^{-1}]_1} \end{bmatrix} Y_t \\ &= \begin{bmatrix} \lambda_1^{P-1} [V^{-1}]_1 Y_t \\ \overline{\lambda_1^{P-1}}\,\overline{[V^{-1}]_1} Y_t \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1^{P-1} [V^{-1}]_1 Y_t \\ \overline{\lambda_1^{P-1} [V^{-1}]_1 Y_t} \end{bmatrix} \end{aligned} \tag{122}$$

When λ contains more than a single set of complex conjugate eigenvalue pairs, the procedure above is applied to each eigenvalue pair $(\lambda_k, \lambda_{k+1})$. In particular, the permutation matrix A will contain A_2 at each $(k, k+1)$ block entry, the $(k, k+1)$ columns of complex conjugate eigenvectors in V will be associated with complex conjugate entries in the $(k, k+1)$ rows of V^{-1}, and each $(X_{k,t}, X_{k+1,t})$ pair will be:

$$\begin{bmatrix} X_{k,t} \\ X_{k+1,t} \end{bmatrix} = \begin{bmatrix} \lambda_k^{P-1} [V^{-1}]_1 Y_t \\ \overline{\lambda_k^{P-1} [V^{-1}]_1 Y_t} \end{bmatrix} \tag{123}$$

■

C.2 AR(2) component of an AR(P) forecast/IRF

Proof of Proposition 5. Re-writing the sum associated with a complex conjugate pairs proceeds in the reverse of the AR(2) exposition in section A.1. Hence:

$$
\begin{aligned}
& \lambda_k^h X_{k,t} + \lambda_{k+1}^h X_{k+1,t} \\
= & \begin{bmatrix} \lambda_k^h & \lambda_{k+1}^h \end{bmatrix} \begin{bmatrix} X_{k,t} \\ X_{k+1,t} \end{bmatrix} \\
= & \begin{bmatrix} \lambda_k & \lambda_{k+1} \end{bmatrix} \begin{bmatrix} \lambda_k^h & 0 \\ 0 & \lambda_{k+1}^h \end{bmatrix} \begin{bmatrix} \lambda_k^{-1} & 0 \\ 0 & \lambda_{k+1}^{-1} \end{bmatrix} \begin{bmatrix} X_{k,t} \\ X_{k+1,t} \end{bmatrix} \\
= & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_k^h & 0 \\ 0 & \lambda_{k+1}^h \end{bmatrix} \begin{bmatrix} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{bmatrix}^{-1} \\
& \times \begin{bmatrix} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_k^{-1} & 0 \\ 0 & \lambda_{k+1}^{-1} \end{bmatrix} \begin{bmatrix} X_{k,t} \\ X_{k+1,t} \end{bmatrix} \\
= & \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{bmatrix}^h \begin{bmatrix} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{bmatrix}^{-1} \right) \begin{bmatrix} 1 & 1 \\ \lambda_k^{-1} & \lambda_{k+1}^{-1} \end{bmatrix} \begin{bmatrix} X_{k,t} \\ X_{k+1,t} \end{bmatrix} \\
= & \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_k & 0 \\ 0 & \lambda_{k+1} \end{bmatrix} \begin{bmatrix} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{bmatrix}^{-1} \right)^h \begin{bmatrix} X_{k,t} + X_{k+1,t} \\ \lambda_k^{-1} X_{k,t} + \lambda_{k+1}^{-1} X_{k+1,t} \end{bmatrix} \\
= & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_k^* & \phi_{k+1}^* \\ 1 & 0 \end{bmatrix}^h \begin{bmatrix} X_{k,t} + X_{k+1,t} \\ \lambda_k^{-1} X_{k,t} + \lambda_{k+1}^{-1} X_{k+1,t} \end{bmatrix}
\end{aligned} \tag{124}
$$

where the final line uses the result:

$$
\begin{aligned}
\begin{bmatrix} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_k & 0 \\ 0 & \lambda_{k+1} \end{bmatrix} \begin{bmatrix} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{bmatrix}^{-1} & = \begin{bmatrix} \lambda_k + \lambda_{k+1} & -\lambda_k \lambda_{k+1} \\ 1 & 0 \end{bmatrix} \\
& = \begin{bmatrix} \phi_k^* & \phi_{k+1}^* \\ 1 & 0 \end{bmatrix}
\end{aligned} \tag{125}
$$

with $\left(\phi_k^*, \phi_{k+1}^*\right) = (\lambda_k + \lambda_{k+1}, -\lambda_k \lambda_{k+1})$.

The expressions above apply to pairs of real or complex conjugate eigenvalues. In the latter case, setting $\lambda_{k+1} = \overline{\lambda_k}$ gives the following:

$$
\begin{aligned}
\begin{bmatrix} X_{k,t} + X_{k+1,t} \\ \lambda_k^{-1} X_{k,t} + \lambda_{k+1}^{-1} X_{k+1,t} \end{bmatrix} & = \begin{bmatrix} X_{k,t} + \overline{X_{k,t}} \\ \lambda_k^{-1} X_{k,t} + \overline{\lambda_k^{-1}} \overline{X_{k,t}} \end{bmatrix} \\
& = \begin{bmatrix} 2 \operatorname{Re}(X_{k,t}) \\ \lambda_k^{-1} X_{k,t} + \overline{\lambda_k^{-1} X_{k,t}} \end{bmatrix} \\
& = \begin{bmatrix} 2 \operatorname{Re}(X_{k,t}) \\ 2 \operatorname{Re}\left(\lambda_k^{-1} X_{k,t}\right) \end{bmatrix}
\end{aligned} \tag{126}
$$

and:

$$
\begin{aligned}
\begin{bmatrix} \phi_k^* & \phi_{k+1}^* \\ 1 & 0 \end{bmatrix} & = \begin{bmatrix} \lambda_k + \overline{\lambda_k} & -\lambda_k \overline{\lambda_k} \\ 1 & 0 \end{bmatrix} \\
& = \begin{bmatrix} 2 \operatorname{Re}(\lambda_k) & -|\lambda_k|^2 \\ 1 & 0 \end{bmatrix}
\end{aligned} \tag{127}
$$

Therefore:

$$\lambda_k^h X_{k,t} + \lambda_{k+1}^h X_{k+1,t} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2\,\mathrm{Re}(\lambda_k) & -|\lambda_k|^2 \\ 1 & 0 \end{bmatrix}^h \begin{bmatrix} 2\,\mathrm{Re}(X_{k,t}) \\ 2\,\mathrm{Re}\left(\lambda_k^{-1} X_{k,t}\right) \end{bmatrix} \tag{128}$$

which is the forecast/IRF expression for an AR(2). ■

As an aside, the last expression in the proof implies that $[X_{1,t}, X_{2,t}]'$ for the actual AR(2) in section A.1 with a complex conjugate pair of eigenvalues should give $[y_t, y_{t-1}]'$, i.e.:

$$\begin{bmatrix} 2\,\mathrm{Re}(X_{1,t}) \\ 2\,\mathrm{Re}\left(\lambda_1^{-1} X_{1,t}\right) \end{bmatrix} = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} \tag{129}$$

The equality $2\,\mathrm{Re}(X_{1,t}) = y_t$ is readily apparent using $X_{1,t}$ from the first row of equation 73, given the imaginary terms sum to zero, hence:

$$\begin{aligned} 2\,\mathrm{Re}(X_{1,t}) &= \frac{2}{2\,\mathrm{Im}(\lambda_1)} \mathrm{Im}(\lambda_1)\, y_t \\ &= y_t \end{aligned} \tag{130}$$

The equality $2\,\mathrm{Re}\left(\lambda_1^{-1} X_{1,t}\right) = y_{t-1}$ is not so apparent, but it may be obtained by directly evaluating $\lambda_1^{-1} X_{1,t}$ with the substitution $\lambda_1^{-1} = \overline{\lambda_1} / |\lambda_1|^2$, i.e.:

$$\begin{aligned} \lambda_1^{-1} X_{1,t} &= \frac{1}{2\,\mathrm{Im}(\lambda_1)} \frac{\overline{\lambda_1}}{|\lambda_1|^2} \left\{ \mathrm{Im}(\lambda_1)\, y_t - i\left[\mathrm{Re}(\lambda_1)\, y_t - |\lambda_1|^2 y_{t-1}\right] \right\} \\ &= \frac{1}{2\,\mathrm{Im}(\lambda_1)} \frac{\mathrm{Re}(\lambda_1) - i\,\mathrm{Im}(\lambda_1)}{|\lambda_1|^2} \\ &\quad \times \left\{ \mathrm{Im}(\lambda_1)\, y_t - i\left[\mathrm{Re}(\lambda_1)\, y_t - |\lambda_1|^2 y_{t-1}\right] \right\} \end{aligned} \tag{131}$$

Expanding this $\lambda_1^{-1} X_{1,t}$ product expression and retaining just the real terms then gives:

$$\begin{aligned} 2\,\mathrm{Re}\left(\lambda_1^{-1} X_{1,t}\right) &= \frac{2}{2\,\mathrm{Im}(\lambda_1)} \frac{1}{|\lambda_1|^2} \\ &\quad \times \left\{ \left(\mathrm{Re}(\lambda_1)\,\mathrm{Im}(\lambda_1)\, y_t - \mathrm{Im}(\lambda_1)\left[\mathrm{Re}(\lambda_1)\, y_t - |\lambda_1|^2 y_{t-1}\right] \right) \right\} \\ &= \frac{1}{\mathrm{Im}(\lambda_1)} \frac{1}{|\lambda_1|^2} \left(\mathrm{Im}(\lambda_1)\, |\lambda_1|^2 y_{t-1} \right) \\ &= y_{t-1} \end{aligned} \tag{132}$$

C.3 AR(2) component forecasts/IRFs in trigonometric form

The forecasts/IRFs for an AR(2) component with complex conjugate eigenvalues may also be expressed in trigonometric form, analogous to the approach for the AR(2) itself in section A.3 of appendix A. That is, setting $(\lambda_1, \lambda_1) = r \exp(\pm i\theta)$ gives:

$$\begin{aligned} &\lambda_k^h X_{k,t} + \lambda_{k+1}^h X_{k+1,t} \\ &= \begin{bmatrix} \lambda_k^h & \overline{\lambda_k^h} \end{bmatrix} \begin{bmatrix} X_{k,t} \\ \overline{X_{k,t}} \end{bmatrix} \\ &= r^h \begin{bmatrix} \exp(ih\theta) & \exp(-ih\theta) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(X_{k,t}) + i\,\mathrm{Im}(X_{k,t}) \\ \mathrm{Re}(X_{k,t}) - i\,\mathrm{Im}(X_{k,t}) \end{bmatrix} \end{aligned} \tag{133}$$

and $\exp(ih\theta)$ may be converted to trigonometric form as in section A.3 using $\exp(ih\theta) = \cos(h\theta) + i\sin(h\theta)$, with the result:

$$\begin{aligned}
& \lambda_k^h X_{k,t} + \lambda_{k+1}^h X_{k+1,t} \\
= \; & r^h \left[\begin{array}{cc} \cos(h\theta) + i\sin(h\theta) & \cos(h\theta) - i\sin(h\theta) \end{array} \right] \left[\begin{array}{c} \mathrm{Re}(X_{k,t}) + i\,\mathrm{Im}(X_{k,t}) \\ \mathrm{Re}(X_{k,t}) - i\,\mathrm{Im}(X_{k,t}) \end{array} \right] \\
= \; & 2r^h \left[\mathrm{Re}(X_{k,t}) \cos(h\theta) - \mathrm{Im}(X_{k,t}) \sin(h\theta) \right] \qquad (134)
\end{aligned}$$

C.4 Historical components as AR(2) data

Proof of Proposition 7. A pair of AR(1) processes from Proposition 4 in section 5.2 is:

$$\left[\begin{array}{c} X_{k,t} \\ X_{k+1,t} \end{array} \right] = \left[\begin{array}{cc} \lambda_k & 0 \\ 0 & \lambda_{k+1} \end{array} \right] \left[\begin{array}{c} X_{k,t-1} \\ X_{k+1,t-1} \end{array} \right] + \left[\begin{array}{c} E_{X,k,t} \\ E_{X,k+1,t} \end{array} \right] \qquad (135)$$

Pre-multiplying by $\left[\begin{array}{cc} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{array} \right]$ gives:

$$\begin{aligned}
& \left[\begin{array}{cc} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{array} \right] \left[\begin{array}{c} X_{k,t} \\ X_{k+1,t} \end{array} \right] \\
= \; & \left[\begin{array}{cc} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{array} \right] \left[\begin{array}{cc} \lambda_k & 0 \\ 0 & \lambda_{k+1} \end{array} \right] \left[\begin{array}{c} X_{k,t-1} \\ X_{k+1,t-1} \end{array} \right] \\
& + \left[\begin{array}{cc} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{array} \right] \left[\begin{array}{c} E_{X,k,t} \\ E_{X,k+1,t} \end{array} \right] \qquad (136)
\end{aligned}$$

and:

$$\begin{aligned}
& \left[\begin{array}{cc} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{array} \right] \left[\begin{array}{cc} \lambda_k & 0 \\ 0 & \lambda_{k+1} \end{array} \right] \left[\begin{array}{c} X_{k,t-1} \\ X_{k+1,t-1} \end{array} \right] \\
= \; & \left[\begin{array}{cc} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{array} \right] \left[\begin{array}{cc} \lambda_k & 0 \\ 0 & \lambda_{k+1} \end{array} \right] \left[\begin{array}{cc} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{array} \right]^{-1} \left[\begin{array}{cc} \lambda_k & \lambda_{k+1} \\ 1 & 1 \end{array} \right] \left[\begin{array}{c} X_{k,t-1} \\ X_{k+1,t-1} \end{array} \right] \\
= \; & \left[\begin{array}{cc} \phi_k^* & \phi_{k+1}^* \\ 1 & 0 \end{array} \right] \left[\begin{array}{c} \lambda_k X_{k,t-1} + \lambda_{k+1} X_{k+1,t-1} \\ X_{k,t-1} + X_{k+1,t-1} \end{array} \right] \qquad (137)
\end{aligned}$$

Using this result and evaluation of the first and third lines of equation 136 gives:

$$\begin{aligned}
& \left[\begin{array}{c} \lambda_k X_{k,t} + \lambda_{k+1} X_{k+1,t} \\ X_{k,t} + X_{k+1,t} \end{array} \right] \\
= \; & \left[\begin{array}{cc} \phi_k^* & \phi_{k+1}^* \\ 1 & 0 \end{array} \right] \left[\begin{array}{c} \lambda_k X_{k,t-1} + \lambda_{k+1} X_{k+1,t-1} \\ X_{k,t-1} + X_{k+1,t-1} \end{array} \right] \\
& + \left[\begin{array}{c} \lambda_k E_{X,k,t} + \lambda_{k+1} E_{X,k+1,t} \\ E_{X,k,t} + E_{X,k+1,t} \end{array} \right] \qquad (138)
\end{aligned}$$

Using the first elements for each of the vector expressions above gives:

$$\begin{aligned}
& \lambda_k X_{k,t} + \lambda_{k+1} X_{k+1,t} \\
= \; & \left[\begin{array}{cc} \phi_k^* & \phi_{k+1}^* \end{array} \right] \left[\begin{array}{c} \lambda_k X_{k,t-1} + \lambda_{k+1} X_{k+1,t-1} \\ X_{k,t-1} + X_{k+1,t-1} \end{array} \right] + \lambda_k E_{X,k,t} + \lambda_{k+1} E_{X,k+1,t} \qquad (139)
\end{aligned}$$

and the second element may be used to complete the regression form of the AR(2). That is:

$$X_{k,t} + X_{k+1,t} = \lambda_k X_{k,t-1} + \lambda_{k+1} X_{k+1,t-1} + E_{X,k,t} + E_{X,k+1,t} \quad (140)$$

and applying the lag operator to both sides gives:

$$X_{k,t-1} + X_{k+1,t-1} = \lambda_k X_{k,t-2} + \lambda_{k+1} X_{k+1,t-2} + E_{X,k,t-1} + E_{X,k+1,t-1} \quad (141)$$

Substituting the expression for $X_{k,t-1} + X_{k+1,t-1}$ into equation 139 gives:

$$\begin{aligned} & \lambda_k X_{k,t} + \lambda_{k+1} X_{k+1,t} \\ = & \begin{bmatrix} \phi_k^* & \phi_{k+1}^* \end{bmatrix} \begin{bmatrix} \lambda_k X_{k,t-1} + \lambda_{k+1} X_{k+1,t-1} \\ \lambda_k X_{k,t-2} + \lambda_{k+1} X_{k+1,t-2} + E_{X,k,t-1} + E_{X,k+1,t-1} \end{bmatrix} \\ & + \lambda_k E_{X,k,t} + \lambda_{k+1} E_{X,k+1,t} \\ = & \begin{bmatrix} \phi_k^* & \phi_{k+1}^* \end{bmatrix} \begin{bmatrix} \lambda_k X_{k,t-1} + \lambda_{k+1} X_{k+1,t-1} \\ \lambda_k X_{k,t-2} + \lambda_{k+1} X_{k+1,t-2} \end{bmatrix} \\ & + \lambda_k E_{X,k,t} + \lambda_{k+1} E_{X,k+1,t} + \phi_{k+1}^* \left(E_{X,k,t-1} + E_{X,k+1,t-1} \right) \end{aligned} \quad (142)$$

The expressions above apply to pairs of real or complex conjugate eigenvalues. In the latter case, setting $\lambda_{k+1} = \overline{\lambda_k}$ and $X_{k+1,t} = \overline{X_{k,t}}$ gives:

$$\begin{aligned} 2\,\mathrm{Re}\left(\lambda_k X_{k,t}\right) & = \begin{bmatrix} \phi_k^* & \phi_{k+1}^* \end{bmatrix} \begin{bmatrix} 2\,\mathrm{Re}\left(\lambda_k X_{k,t-1}\right) \\ 2\,\mathrm{Re}\left(\lambda_k X_{k,t-2}\right) \end{bmatrix} \\ & \quad + 2\,\mathrm{Re}\left(\lambda_k E_{X,k,t}\right) + \phi_{k+1}^* 2\,\mathrm{Re}\left(E_{X,k,t-1}\right) \end{aligned} \quad (143)$$

■

D Additional material and proofs for section 5.4

This appendix first provides, in section D.1,, background on the summation expression for forecast error variances (FEVs). Section D.2 provides the proof of the closed-form expression for the FEV summation in Proposition 9, which makes use of a supplementary proposition and proof at the start of the section. Section D.3 provides the proof of the closed-form expression for the ergodic variance in Proposition 10. Analytic examples of the FEV and ergodic variances for the AR(1) and AR(2) models are provided in section D.4.

D.1 Forecast errors and FEVs

The forecast error for a given horizon H is defined as:

$$y_{t+H} - \mathbb{E}_t\left[y_{t+H}\right] = J\left(Y_{t+H} - \mathbb{E}_t\left[Y_{t+H}\right]\right) \quad (144)$$

and $y_{t+H} - \mathbb{E}_t\left[y_{t+H}\right]$ requires the contribution from each period h up to horizon H, i.e.:

$$\begin{aligned} Y_{t+1} - \mathbb{E}_t\left[Y_{t+1}\right] & = E_{Y,t+1} \\ Y_{t+2} - \mathbb{E}_t\left[Y_{t+2}\right] & = \Phi E_{Y,t+1} + E_{Y,t+2} \\ & \vdots \\ Y_{t+H} - \mathbb{E}_t\left[Y_{t+H}\right] & = \sum_{h=0}^{H-1} \Phi^h E_{Y,t+H-h} \end{aligned} \quad (145)$$

The FEV is obtained from the expected value of the forecast error Y_{t+H}, i.e.:

$$\begin{aligned}
\Omega_Y(H) &= \mathbb{E}_t\left\{(Y_{t+H}-\mathbb{E}_t[Y_{t+H}])(Y_{t+H}-\mathbb{E}_t[Y_{t+H}])'\right\} \\
&= \mathbb{E}_t\left\{\left(\sum_{h=0}^{H-1}\Phi^h E_{Y,t+H-h}\right)\left(\sum_{h=0}^{H-1}\Phi^h E_{Y,t+H-h}\right)'\right\} \\
&= \mathbb{E}_t\left\{\left(\sum_{h=0}^{H-1}\Phi^h E_{Y,t+H-h}\right)\left(\sum_{h=0}^{H-1}E'_{Y,t+H-h}\left(\Phi^h\right)'\right)\right\} \\
&= \sum_{h=0}^{H-1}\Phi^h\mathbb{E}_t\left[E_{Y,t+H-h}E'_{Y,t+H-h}\right]\left(\Phi^h\right)' \qquad (146)
\end{aligned}$$

where the last line retains only the $E_{Y,t+H-h}E'_{Y,t+H-h}$ terms because the other terms have an expected value of zero, given the assumed properties of the residuals in section 2.1, i.e. $\mathbb{E}_t[\varepsilon_u\varepsilon'_v]=\Omega_\varepsilon$ if $u=v$ and zero otherwise or simply iid normal $\varepsilon_t\sim N(0,\Omega_\varepsilon)$. Specifically, from equation 34, $E_{Y,t}=[\varepsilon_t,0,\ldots,0]'$, so $E_{Y,u}E'_{Y,v}$ is a $P\times P$ matrix with $\varepsilon_u\varepsilon_v$ as the $(1,1)$ element and zeros otherwise. Applying the expectations operator then gives $\mathbb{E}_t[\varepsilon_u\varepsilon_v]=\Omega_\varepsilon$ if $u=v$ and zero otherwise.

Hence, only $\mathbb{E}_t\left[E_{Y,t+H-h}E'_{Y,t+H-h}\right]$ will contain non-zero terms, i.e. $\mathbb{E}_t\left[\varepsilon^2_{t+H-h}\right]=\Omega_\varepsilon$ as the $(1,1)$ element and zeros otherwise, which may also be expressed as $\Omega_{E_Y}=J'\Omega_\varepsilon J$. In summary:

$$\Omega_Y(H)=\sum_{h=0}^{H-1}\Phi^h\Omega_{E_Y}\left(\Phi^h\right)' \qquad (147)$$

The FEV for $\Omega_y(H)$ is obtained from $\Omega_Y(H)$ as:

$$\Omega_y(H)=J\left(\sum_{h=0}^{H-1}\Phi^h\Omega_{E_Y}\left(\Phi^h\right)'\right)J' \qquad (148)$$

Note that my summation expression for $\Omega_y(H)$ is equivalent to the summation expression from Lütkepohl (2006), although some alignment of notation is necessary to make that clear. That is, the verbatim expression from Lütkepohl (2006) p.38, eq. 2.2.11 is:

$$\Sigma_y(h):=\mathrm{MSE}[y_t(h)]=\sum_{i=0}^{h-1}\Phi_i\Sigma_u\Phi'_i=\Sigma_y(h-1)+\Phi_{h-1}\Sigma_u\Phi'_{h-1} \qquad (149)$$

where, from Lütkepohl (2006) equation 2.1.17, $\Phi_i=JA^iJ'$. My notation (on the left-hand side of the following expressions) in terms of the Lütkepohl (2006) notation (on the right-hand side) is $\Omega_y(H)\equiv\Sigma_y(h)$, $\Phi\equiv A$, $\Omega_\varepsilon\equiv\Sigma_u$, $J\Phi^hJ'\equiv\Phi_h$, $H\equiv h$, and $h\equiv i$. Furthermore:

$$J\left(\sum_{h=0}^{H-1}\Phi^h\Omega_{E_Y}\left(\Phi^h\right)'\right)J'=\sum_{h=0}^{H-1}J\Phi^hJ'\Omega_\varepsilon J\left(\Phi^h\right)'J' \qquad (150)$$

The right-hand side expression now replicates the summation expression in Lütkepohl (2006) equation 2.1.17, but in my notation. Note that the subsequent equality in Lütkepohl (2006) equation 2.1.17, i.e. $\sum_{i=0}^{h-1}\Phi_i\Sigma_u\Phi'_i=\Sigma_y(h-1)+\Phi_{h-1}\Sigma_u\Phi'_{h-1}$, is the recursive expression for the FEV, which begins from Σ_u in for horizon $h=1$, and adds increments $\Phi_{h-1}\Sigma_u\Phi'_{h-1}$ for subsequent horizons. My aim to is to produce a closed-form solution for the summation up to the given horizon.

D.2 Closed-form solution for FEV

Proposition D.1 *Given the diagonal matrix $\Lambda^h = diag\left(\left[\lambda_1^h, \ldots, \lambda_P^h\right]\right)$, its Hermitian (or conjugate) transpose $\left(\Lambda^\dagger\right)^h$, and a generic Hermitian matrix U, the individual elements of the matrix product $\Lambda^h U \left(\Lambda^\dagger\right)^h$ may be expressed as:*

$$\left[\Lambda^h U \left(\Lambda^\dagger\right)^h\right]_{ij} = U_{ij} \left(\lambda_i \overline{\lambda_j}\right)^h \tag{151}$$

Proof. When multiplying two generic matrices A and B with a conformable inner dimension of K, the (i, j) element of the result, i.e. $[AB]_{ij}$, is the summation $[AB]_{ij} = \sum_{k=1}^{K} B_{ik} C_{kj}$. The matrix product $\Lambda^h U \Lambda^h$ contains two sets of matrix multiplications, so the index l is used for the $U \left(\Lambda^\dagger\right)^h$ product, and the index k is used for the $\Lambda^h \left[U \left(\Lambda^\dagger\right)^h\right]$ product, i.e.:

$$\begin{aligned}
\left[\Lambda^h U \left(\Lambda^\dagger\right)^h\right]_{ij} &= \sum_{k=1}^{P} \left[\Lambda^h\right]_{ik} \left(\sum_{l=1}^{P} U_{kl} \left[\Lambda^h\right]^\dagger_{lj}\right) \\
&= \sum_{k=1}^{P} \left[\Lambda^h\right]_{ik} U_{kj} \left[\Lambda^h\right]^\dagger_{jj} \\
&= \left[\Lambda^h\right]_{ii} U_{ij} \left[\Lambda^h\right]^\dagger_{jj}
\end{aligned} \tag{152}$$

where $\sum_{l=1}^{P} U_{kl} \left[\Lambda^h\right]^\dagger_{lj} = U_{kj} \left[\Lambda^h\right]^\dagger_{jj}$ because $\left(\Lambda^\dagger\right)^h$ is diagonal, so $\left[\Lambda^h\right]^\dagger_{lj}$ is only non-zero when $l = j$. Similarly, $\sum_{k=1}^{P} \left[\Lambda^h\right]_{ik} U_{kj} \left[\Lambda^h\right]^\dagger_{jj} = \left[\Lambda^h\right]_{ii} U_{ij} \left[\Lambda^h\right]^\dagger_{jj}$ because $\left[\Lambda^h\right]_{ik}$ is only non-zero when $k = i$. Substituting the elements $\left[\Lambda^h\right]_{ii} = \lambda_i$ and $\left[\Lambda^h\right]^\dagger_{jj} = \overline{\lambda_j}$ into $\left[\Lambda^h\right]_{ii} U_{ij} \left[\Lambda^h\right]^\dagger_{jj}$ gives:

$$\begin{aligned}
\left[\Lambda^h U \Lambda^h\right]_{ij} &= \lambda_i^h U_{ij}^h \overline{\lambda_j}^h \\
&= U_{ij} \left(\lambda_i \overline{\lambda_j}\right)^h
\end{aligned} \tag{153}$$

■

Directly evaluating an example with $P = 2$ illustrates the form that $\Lambda^h U \left(\Lambda^h\right)^\dagger$ will take in general:

$$\begin{aligned}
&\begin{bmatrix} \lambda_1^h & 0 \\ 0 & \lambda_2^h \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \overline{\lambda_1}^h & 0 \\ 0 & \overline{\lambda_2}^h \end{bmatrix} \\
= &\begin{bmatrix} \lambda_1^h & 0 \\ 0 & \lambda_2^h \end{bmatrix} \begin{bmatrix} U_{11} \overline{\lambda_1}^h & U_{12} \overline{\lambda_2}^h \\ U_{21} \overline{\lambda_1}^h & U_{22} \overline{\lambda_2}^h \end{bmatrix} \\
= &\begin{bmatrix} \lambda_1^h U_{11} \overline{\lambda_1}^h & \lambda_1^h U_{12} \overline{\lambda_2}^h \\ \lambda_2^h U_{21} \overline{\lambda_1}^h & \lambda_2^h U_{22} \overline{\lambda_2}^h \end{bmatrix} \\
= &\begin{bmatrix} \frac{U_{11} \left(\lambda_1 \overline{\lambda_1}\right)^h}{U_{12} \left(\lambda_1 \overline{\lambda_2}\right)^h} & \begin{matrix} U_{12} \left(\lambda_1 \overline{\lambda_2}\right)^h \\ U_{22} \lambda_2^h \overline{\lambda_2}^h \end{matrix} \end{bmatrix}
\end{aligned} \tag{154}$$

where $U_{21}\overline{\lambda_1}^h\lambda_2^h = \overline{U_{12}\left(\lambda_1\overline{\lambda_2}\right)^h}$. Note that any diagonal matrix could obviously be substituted for Λ^h, but the notation anticipates the context of the proof for Proposition 8.

Note that, for consistency with the notation in Proposition 8, I have based the proof below on the eigensystem decomposition $\Phi = V_X \Lambda V_X^{-1}$, where $V_X = V\Lambda^{1-P}$ with V and Λ from the eigensystem decomposition $\Phi = V\Lambda V^{-1}$. The closed-form FEV and ergodic variance expressions could be equivalently based on the eigensystem decomposition $\Phi = V\Lambda V^{-1}$, in which case $V_X^{-1}\Omega_{E_Y}\left(V_X^{-1}\right)^\dagger$ in the proof of Proposition 8 would be replaced by $V^{-1}\Omega_{E_Y}\left(V^{-1}\right)^\dagger$ and the final calculation would be $\Omega_y(H) = JV\Omega_X(H)V^\dagger J'$. The equivalence is apparent as follows:

$$
\begin{aligned}
V_X \Lambda V_X^{-1} &= \left(V\Lambda^{1-P}\right)\Lambda\left(V\Lambda^{1-P}\right)^{-1} \\
&= V\Lambda^{1-P}\Lambda\Lambda^{P-1}V^{-1} \\
&= V\Lambda V^{-1}
\end{aligned} \tag{155}
$$

Proof of Proposition 8. Begin with the FEV summation expression, i.e.:

$$
\Omega_Y(H) = \sum_{h=0}^{H-1} \Phi^h \Omega_{E_Y}\left(\Phi^h\right)' \tag{156}
$$

and use the decomposition $\Phi = V_X \Lambda V_X^{-1}$ to re-express the summation as follows:

$$
\begin{aligned}
\Omega_Y(H) &= \sum_{h=0}^{H-1} V_X \Lambda^h V_X^{-1}\Omega_{E_Y}\left(V_X\Lambda^h V_X^{-1}\right)^\dagger \\
&= \sum_{h=0}^{H-1} V_X \Lambda^h V_X^{-1}\Omega_{E_Y}\left(V_X^{-1}\right)^\dagger\left(\Lambda^\dagger\right)^h V_X^\dagger \\
&= V_X\left[\sum_{h=0}^{H-1}\Lambda^h\Omega_{E_X}\left(\Lambda^\dagger\right)^h\right]V_X^\dagger
\end{aligned} \tag{157}
$$

where "†"denotes the Hermitian/conjugate transpose, $\Omega_{E_X} = V_X^{-1}\Omega_{E_Y}\left(V_X^{-1}\right)^\dagger$, which is a Hermitian $P \times P$ matrix, i.e.$\Omega_{E_X,j,i} = \overline{\Omega_{E_X,j,i}}$, and I will denote the matrix defined by the summation in the square brackets as $\Omega_X(H)$, i.e.:

$$
\Omega_X(H) = \sum_{h=0}^{H-1}\Lambda^h\Omega_{E_X}\left(\Lambda^\dagger\right)^h \tag{158}
$$

From Proposition 12, each (i, j) element in the summation for $\Omega_X(H)$ is therefore the following:

$$
\begin{aligned}
\left[\Omega_X(H)\right]_{ij} &= \left[\sum_{h=0}^{H-1} \Lambda^h \Omega_{E_X} \left(\Lambda^{\dagger}\right)^h\right]_{ij} \\
&= \sum_{h=0}^{H-1} \Omega_{E_X,ij} \left(\lambda_i \overline{\lambda_j}\right)^h \\
&= \Omega_{E_X,ij} \sum_{h=0}^{H-1} \left(\lambda_i \overline{\lambda_j}\right)^h \\
&= \Omega_{E_X,ij} \frac{1-\left(\lambda_i \overline{\lambda_j}\right)^H}{1-\lambda_i \overline{\lambda_j}}
\end{aligned} \tag{159}
$$

where the summation of the geometric series $\left(\lambda_i \overline{\lambda_j}\right)^h$ in the third line has been replaced with its closed-form analytic result in last line, which requires $\left|\lambda_i \overline{\lambda_j}\right| < 1$. That is, while the sum of a geometric series is well-known, in the present context:

$$
\begin{aligned}
\left(1-\lambda_i \overline{\lambda_j}\right) \sum_{h=0}^{H-1} \left(\lambda_i \overline{\lambda_j}\right)^h &= \left(\sum_{h=0}^{H-1} \left(\lambda_i \overline{\lambda_j}\right)^h\right) - \left[\sum_{h=0}^{H-1} \left(\lambda_i \overline{\lambda_j}\right)^{h+1}\right] \\
&= \left(1+\sum_{h=1}^{H-1} \left(\lambda_i \overline{\lambda_j}\right)^h\right) - \left[\sum_{h=1}^{H} \left(\lambda_i \overline{\lambda_j}\right)^h\right] \\
&= \left(1+\sum_{h=1}^{H-1} \left(\lambda_i \overline{\lambda_j}\right)^h\right) - \left[\left(\lambda_i \lambda_j\right)^H + \sum_{h=1}^{H-1} \left(\lambda_i \overline{\lambda_j}\right)^h\right] \\
&= 1-\left(\lambda_i \overline{\lambda_j}\right)^H \\
\sum_{h=0}^{H-1} \left(\lambda_i \overline{\lambda_j}\right)^h &= \frac{1-\left(\lambda_i \overline{\lambda_j}\right)^H}{1-\lambda_i \overline{\lambda_j}}
\end{aligned} \tag{160}
$$

Each (i, j) element of $\Omega_X(H)$ may therefore be calculated using the closed-form solution given in equation 159, although only $P(P+1)/2$ unique calculations are required due to its symmetry. The FEV for an arbitrary horizon H is then:

$$
\Omega_Y(H) = V_X \Omega_X(H) V_X^{\dagger} \tag{161}
$$

and the FEV for $\Omega_y(H)$ is:

$$
\begin{aligned}
\Omega_y(H) &= J \Omega_Y(H) J' \\
&= J V_X \Omega_X(H) V_X^{\dagger} J'
\end{aligned} \tag{162}
$$

■

D.3 Closed-form solution for ergodic variance

The section presents the proof for Proposition 9, and then further discusses the ergodic variance result in the context of the literature.

Proof of Proposition 9. From Lütkepohl (2006) eq. 2.1.18, the ergodic variance for y_t, which I will denote $\Omega_y(\infty)$, is:

$$\Omega_y(\infty) = \sum_{h=0}^{\infty} J\Phi^h \Omega_{E_Y} \left(\Phi^h\right)' J' \tag{163}$$

Using the results from the previous section, the closed-form expression for this infinite sum is:

$$\begin{aligned}
\Omega_y(\infty) &= \lim_{H\to\infty} [\Omega_y(H)] \\
&= \lim_{H\to\infty} \left[JV\Omega_X(H) V^\dagger J'\right] \\
&= JV\Omega_X(\infty) V^\dagger J'
\end{aligned} \tag{164}$$

where:

$$\begin{aligned}
\Omega_X(\infty) &= \lim_{H\to\infty} [\Omega_X(H)] \\
&= \sum_{h=0}^{\infty} \Lambda^h \Omega_{E_X} \left(\Lambda^\dagger\right)^h
\end{aligned} \tag{165}$$

and the elements of $\Omega_X(\infty)$ are obtained as:

$$\begin{aligned}
[\Omega_X(\infty)]_{ij} &= \lim_{H\to\infty} \left(\Omega_{E_X,ij} \frac{1-\left(\lambda_i\overline{\lambda_j}\right)^H}{1-\lambda_i\overline{\lambda_j}}\right) \\
&= \Omega_{E_X,ij} \frac{1}{1-\lambda_i\overline{\lambda_j}}
\end{aligned} \tag{166}$$

■

The ergodic variance is also the solution to the discrete-time Lyapunov equation, which in the present context may be expressed as $\Phi\Omega_y(\infty)\Phi + \Omega_{E_X} = \Omega_y(\infty)$, where all eigenvalues of Φ have magnitudes less than 1. As discussed in Doan (2010), there are a variety of methods for solving the Lyapunov equation. The method based on vectorization is often presented in econometrics textbooks, i.e.:[29]

$$\text{vec}[\Omega_Y(\infty)] = (I_{P^2} - \Phi \otimes \Phi)^{-1} \text{vec}(\Omega_{E_Y}) \tag{167}$$

Doan (2010) notes that the inversion of the $P^2 \times P^2$ matrix requires $O(P^6)$ arithmetic operations, while methods that retain the original matrix dimensions, e.g. Kitagawa (1977) and Johannsen (2002), require just $O(P^3)$ operations. The method developed in the present paper is also $O(P^3)$, so is within the class of efficient class of Lyapunov equation solutions, but the relative advantage is the intuition of the result being the infinite limit of the FEV summation. Note also that my finite-horizon $\Omega_Y(H)$ and infinite-horizon results $\Omega_Y(\infty)$ are the discrete-time analogue of the continuous-time expression derived in Rome (1969).

[29] See, for example, Hamilton (1994) p. 265 and Lütkepohl (2006) eq. 2.1.39.

In light of the FEV and ergodic variance results, the FEV $\Omega_X(H)$ may be equivalently expressed as $\Omega_X(\infty)$ and $\Omega_X(H)$ relative to $\Omega_X(\infty)$, i.e.

$$\Omega_X(H) = \Omega_X(\infty) + [\Omega_X(H) - \Omega_X(\infty)] \tag{168}$$

where the elements of $[\Omega_X(H) - \Omega_X(\infty)]$ are:

$$[\Omega_X(H)]_{ij} - [\Omega_X(\infty)]_{ij} = -\Omega_{E_X,ij}\frac{\left(\lambda_i\overline{\lambda_j}\right)^H}{1-\lambda_i\overline{\lambda_j}} \tag{169}$$

Therefore, regardless of the method used to obtain the ergodic variance, the result may be used with the adjustment $[\Omega_X(H) - \Omega_X(\infty)]$ to obtain the FEV.

As an aside, the vectorization method to be more efficient than $O(P^6)$ noted by Doan (2010) in the case of an AR(P), if the $P \times P$ eigensystem decomposition $\Phi = VDV^{-1}$ and the diagonal form of D are exploited, along with only the first element in vec(Ω_{E_Y}) being non-zero, i.e. vec$[\Omega_Y] = [\Omega_\varepsilon, 0, \ldots, 0]$. Specifically, $\Phi \otimes \Phi$ may be re-expressed as:

$$\begin{aligned}\Phi \otimes \Phi &= \left(VDV^{-1}\right) \otimes \left(VDV^{-1}\right) \\ &= (V \otimes V)(D \otimes D)\left(V^{-1} \otimes V^{-1}\right)\end{aligned} \tag{170}$$

where the latter result follows from the mixed product property of the Kronecker product, i.e. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ where A, B, C, and D are generic matrices that are appropriately conformable. Therefore, with the additional generic matrices E and F:

$$\begin{aligned}(A \otimes B)(C \otimes D)(E \otimes F) &= [(AC) \otimes (BD)](E \otimes F) \\ &= (ACE) \otimes (BDF)\end{aligned} \tag{171}$$

The matrix $I_{P^2} - \Phi \otimes \Phi$ to be inverted may therefore be written as:

$$\begin{aligned}I_{P^2} - \Phi \otimes \Phi &= I_{P^2} - (V \otimes V)(D \otimes D)\left(V^{-1} \otimes V^{-1}\right) \\ &= (V \otimes V) I_{P^2}\left(V^{-1} \otimes V^{-1}\right) - (V \otimes V)(D \otimes D)\left(V^{-1} \otimes V^{-1}\right) \\ &= (V \otimes V)(I_{P^2} - D \otimes D)\left(V^{-1} \otimes V^{-1}\right)\end{aligned} \tag{172}$$

and therefore:

$$\begin{aligned}[I_{P^2} - \Phi \otimes \Phi]^{-1} &= \left[(V \otimes V)(I_{P^2} - D \otimes D)\left(V^{-1} \otimes V^{-1}\right)\right]^{-1} \\ &= \left(V^{-1} \otimes V^{-1}\right)^{-1}(I_{P^2} - D \otimes D)(V \otimes V)^{-1} \\ &= (V \otimes V)(I_{P^2} - D \otimes D)^{-1}\left(V^{-1} \otimes V^{-1}\right)\end{aligned} \tag{173}$$

The vectorization expression therefore becomes:

$$\text{vec}[\Omega_Y(\infty)] = (V \otimes V)(I_{P^2} - D \otimes D)^{-1}\left(V^{-1} \otimes V^{-1}\right)\text{vec}(\Omega_{E_Y}) \tag{174}$$

With D diagonal, $D \otimes D$ and $I_{P^2} - D \otimes D$ are diagonal, and so inverting the latter requires just the reciprocal of $2P$ elements. Also, only the diagonal elements of $V \otimes V$ are required for the product $(V \otimes V)(I_{P^2} - D \otimes D)^{-1}$. Because only the first element in vec(Ω_{E_Y}) is non-zero for an AR(P), only the first row in $V^{-1} \otimes V^{-1}$ needs to be obtained when calculating $(V^{-1} \otimes V^{-1})$vec$[\Omega_Y]$ in the vectorization expression.

D.4 FEV and ergodic variance for the AR(1) and AR(2)

This section contains analytic examples of the closed-form variances for the AR(1) and AR(2) models. Unlike higher-order AR(P) models that would need to be calculated numerically in practice, the AR(1) has very succinct algebraic solutions, and the AR(2) solutions are relatively succinct. The results for the AR(1) and AR(2) ergodic variances may be compared to those already available in the literature, as noted below. However, I'm not aware of closed-form results for the FEVs of the AR(1) and AR(2).

D.4.1 AR(1) example

The AR(1) provides trivial analytic example, given $\phi_1 = \lambda_1$ and the residual variances are $\Omega_\varepsilon = \Omega_{E_Y} = \Omega_{E_X}$. Therefore all methods produce identical results for the ergodic variances $\Omega_y(\infty) = \Omega_Y(\infty) = \Omega_X(\infty)$, i.e.:

$$\Omega_y(\infty) = \frac{\Omega_\varepsilon}{1-\phi_1^2} \tag{175}$$

which matches Hamilton (1994) p. 58.

The FEV for the AR(1) is:

$$\Omega_y(H) = \frac{\Omega_\varepsilon}{1-\phi_1^2}\left(1-\phi_1^{2H}\right) \tag{176}$$

and $\Omega_y(H) = \Omega_Y(H) = \Omega_X(H)$.

D.4.2 AR(2) example with distinct eigenvalues

The ergodic variance for the AR(2) is well-known, e.g. from Hamilton (1994) p. 58:

$$\Omega_y(\infty) = \frac{(1-\phi_2)}{(1+\phi_2)\left[(1-\phi_2)^2-\phi_1^2\right]}\Omega_\varepsilon \tag{177}$$

so it serves as a useful analytic example to compare the different calculation methods. For use further below, I use the substitutions $(\phi_1, \phi_2) = (\lambda_1+\lambda_2, -\lambda_1\lambda_2)$ from equation 7 to express $\Omega_y(\infty)$ in terms of the AR(2) eigenvalues , i.e.:

$$\begin{aligned}
\Omega_y(\infty) &= \frac{1+\lambda_1\lambda_2}{(1-\lambda_1\lambda_2)\left[(1+\lambda_1\lambda_2)^2-(\lambda_1+\lambda_2)^2\right]}\Omega_\varepsilon \\
&= \frac{1+\lambda_1\lambda_2}{(1-\lambda_1\lambda_2)\left(1+\lambda_1^2\lambda_2^2-\lambda_1^2-\lambda_2^2\right)}\Omega_\varepsilon \\
&= \frac{1+\lambda_1\lambda_2}{(1-\lambda_1\lambda_2)\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right)}\Omega_\varepsilon
\end{aligned} \tag{178}$$

The calculation of $\Omega_y(\infty)$ using the vectorized expression in equation 167 requires the inversion of a 4×4 matrix, i.e.:

$$\begin{aligned}
\mathrm{vec}\left[\Omega_Y(\infty)\right] &= \left(I_{P^2} - \Phi \otimes \Phi\right)^{-1} \mathrm{vec}\left[\Omega_{E_Y}\right] \\
&= \left(I_4 - \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} \Omega_\varepsilon \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \phi_1^2 & \phi_1\phi_2 & \phi_1\phi_2 & \phi_2^2 \\ \phi_1 & 0 & \phi_2 & 0 \\ \phi_1 & \phi_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} \Omega_\varepsilon \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{1 - \phi_1^2\phi_2 - \phi_1^2 + \phi_2^3 - \phi_2^2 - \phi_2} \begin{bmatrix} 1 - \phi_2 \\ \phi_1 \\ \phi_1 \\ 1 - \phi_2 \end{bmatrix} \Omega_\varepsilon \\
&= \frac{1}{(1 + \phi_2)\left[(1 - \phi_2)^2 - \phi_1^2\right]} \begin{bmatrix} 1 - \phi_2 \\ \phi_1 \\ \phi_1 \\ 1 - \phi_2 \end{bmatrix} \Omega_\varepsilon
\end{aligned} \tag{179}$$

Re-constituting the matrix from vec$[\Omega_Y(\infty)]$ gives $\Omega_Y(\infty)$, i.e.:

$$\Omega_Y(\infty) = \frac{\Omega_\varepsilon}{(1 + \phi_2)\left[(1 - \phi_2)^2 - \phi_1^2\right]} \begin{bmatrix} 1 - \phi_2 & \phi_1 \\ \phi_1 & 1 - \phi_2 \end{bmatrix} \tag{180}$$

The ergodic variance $\Omega_y(\infty)$ is the top-left element, i.e.:

$$\begin{aligned}
\Omega_y(\infty) &= J'\Omega_Y(\infty) J \\
&= \frac{\Omega_\varepsilon}{(1 + \phi_2)\left[(1 - \phi_2)^2 - \phi_1^2\right]} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 - \phi_2 & \phi_1 \\ \phi_1 & 1 - \phi_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1 - \phi_2}{(1 + \phi_2)\left[(1 - \phi_2)^2 - \phi_1^2\right]} \Omega_\varepsilon
\end{aligned} \tag{181}$$

which matches equation 177.

The calculation of $\Omega_y(\infty)$ based on the the AR(2) eigensystem uses the AR(2) decomposition $\Phi = V\Lambda V^{-1}$ from section A.1 of appendix A, i.e.:

$$\begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \tag{182}$$

Rather than using $\Omega_{E_X} = V_X^{-1}\Omega_{E_Y}\left(V_X^{-1}\right)^\dagger$ and $\Omega_y(\infty) = JV_X\Omega_X(H)V_X^\dagger J'$ as defined in the Proposition 8 and its proof, which is based on $\Phi = V_X\Lambda V_X^{-1}$ and $X = \Lambda^{P-1}V^{-1}Y_t$, in this section I will use $\Omega_{E_X}^* = V^{-1}\Omega_{E_Y}\left(V^{-1}\right)^\dagger$ and $\Omega_y(\infty) = JV\Omega_X^*(\infty)V^\dagger J'$, which is based on $\Phi = V\Lambda V^{-1}$ and $X_t^* = V^{-1}Y_t$. Hence:

$$\begin{aligned}
\Omega_{E_X}^* &= V^{-1}\Omega_{E_Y}\left(V^{-1}\right)^\dagger \\
&= \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} \Omega_\varepsilon & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\overline{\lambda_1-\lambda_2}}\begin{bmatrix} 1 & -1 \\ -\overline{\lambda_2} & \overline{\lambda_1} \end{bmatrix} \\
&= \frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -\overline{\lambda_2} & \overline{\lambda_1} \end{bmatrix} \\
&= \frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} && (183)
\end{aligned}$$

As an aside, an expression for $\Omega_X^*(\infty)$ may be calculated directly using the vectorized expression, i.e.:

$$\begin{aligned}
& \operatorname{vec}\left[\Omega_X^*(\infty)\right] \\
=\ & \left(I_{P^2} - \Lambda\otimes\Lambda^\dagger\right)^{-1}\operatorname{vec}\left[\Omega_{E_X}^*\right] && (184) \\
=\ & \left(I_4 - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\otimes\begin{bmatrix} \overline{\lambda_1} & 0 \\ 0 & \overline{\lambda_2} \end{bmatrix}\right)^{-1}\frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \\
=\ & \frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\left(\begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} - \begin{bmatrix} |\lambda_1|^2 & 0 & 0 & 0 \\ 0 & \lambda_1\overline{\lambda_2} & 0 & 0 \\ 0 & 0 & \overline{\lambda_1}\lambda_2 & 0 \\ 0 & 0 & 0 & |\lambda_2|^2 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \\
=\ & \frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\begin{bmatrix} \frac{1}{1-|\lambda_1|^2} \\ -\frac{1}{1-\lambda_1\overline{\lambda_2}} \\ -\frac{1}{1-\overline{\lambda_1}\lambda_2} \\ \frac{1}{1-|\lambda_2|^2} \end{bmatrix} && (185)
\end{aligned}$$

While this still requires the inversion of a 4×4 matrix, the diagonal form of the Kronecker product that arises from the diagonal eigenvalue matrix makes the inversion very straightforward. Re-creating the matrix gives $\Omega_X^*(\infty)$, i.e.:

$$\Omega_X^*(\infty) = \frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\begin{bmatrix} \frac{1}{1-|\lambda_1|^2} & -\frac{1}{1-\lambda_1\overline{\lambda_2}} \\ -\frac{1}{1-\overline{\lambda_1}\lambda_2} & \frac{1}{1-|\lambda_2|^2} \end{bmatrix} \qquad (186)$$

Calculating the individual elements of $\left[\Omega_X^*(\infty)\right]_{ij} = \Omega_{E_X,ij}^* 1/\left(1-\lambda_i\overline{\lambda_j}\right)$ directly from equation 52 obviously gives the identical result for $\Omega_X^*(\infty)$, but directly as the elements of the 2×2 matrix, i.e.:

$$\begin{aligned}
\Omega_X^*(\infty) &= \Omega_{E_X}^*\circ\begin{bmatrix} \frac{1}{1-|\lambda_1|^2} & \frac{1}{1-\lambda_1\overline{\lambda_2}} \\ \frac{1}{1-\overline{\lambda_1}\lambda_2} & \frac{1}{1-|\lambda_2|^2} \end{bmatrix} \\
&= \frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\circ\begin{bmatrix} \frac{1}{1-|\lambda_1|^2} & \frac{1}{1-\lambda_1\overline{\lambda_2}} \\ \frac{1}{1-\overline{\lambda_1}\lambda_2} & \frac{1}{1-|\lambda_2|^2} \end{bmatrix} \\
&= \frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\begin{bmatrix} \frac{1}{1-|\lambda_1|^2} & -\frac{1}{1-\lambda_1\overline{\lambda_2}} \\ -\frac{1}{1-\overline{\lambda_1}\lambda_2} & \frac{1}{1-|\lambda_2|^2} \end{bmatrix} && (187)
\end{aligned}$$

where "$\circ$" denotes the Hadamard or element-wise product.

Noting that $JV = [\lambda_1, \lambda_2]$ and $V^\dagger J' = \left[\overline{\lambda_1}, \overline{\lambda_2}\right]'$, the eigensystem calculation of $\Omega_y(\infty)$ is then:

$$\begin{aligned}
\Omega_y(\infty) &= JV\Omega_X^*(\infty)\,V^\dagger J' \\
&= \frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\left[\begin{array}{cc}\lambda_1 & \lambda_2\end{array}\right]\left[\begin{array}{cc}\frac{1}{1-|\lambda_1|^2} & -\frac{1}{1-\lambda_1\overline{\lambda_2}} \\ -\frac{1}{1-\overline{\lambda_1}\lambda_2} & \frac{1}{1-|\lambda_2|^2}\end{array}\right]\left[\begin{array}{c}\overline{\lambda_1} \\ \overline{\lambda_2}\end{array}\right] \\
&= \frac{\Omega_\varepsilon}{(\lambda_1-\lambda_2)\left(\overline{\lambda_1}-\overline{\lambda_2}\right)}\left(\frac{1-|\lambda_1|^2|\lambda_2|^2(\lambda_1-\lambda_2)\left(\overline{\lambda_1}-\overline{\lambda_2}\right)}{\left(1-|\lambda_1|^2\right)\left(1-\lambda_1\overline{\lambda_2}\right)\left(1-\overline{\lambda_1}\lambda_2\right)\left(1-|\lambda_2|^2\right)}\right) \\
&= \frac{1-|\lambda_1|^2|\lambda_2|^2}{\left(1-|\lambda_1|^2\right)\left(1-\lambda_1\overline{\lambda_2}\right)\left(1-\overline{\lambda_1}\lambda_2\right)\left(1-|\lambda_2|^2\right)}\Omega_\varepsilon \qquad (188)
\end{aligned}$$

where $|\lambda_1-\lambda_2|^2 = (\lambda_1-\lambda_2)\overline{(\lambda_1-\lambda_2)} = (\lambda_1-\lambda_2)\left(\overline{\lambda_1}-\overline{\lambda_2}\right)$ has been used in the third line.

In the case of real eigenvalues, the numerator is $1-|\lambda_1|^2|\lambda_2|^2 = 1-\lambda_1^2\lambda_2^2 = (1+\lambda_1\lambda_2)\times(1-\lambda_1\lambda_2)$, and in the denominator $\left(1-\lambda_1\overline{\lambda_2}\right)\left(1-\overline{\lambda_1}\lambda_2\right) = (1-\lambda_1\lambda_2)^2$. Therefore:

$$\begin{aligned}
\Omega_y(\infty) &= \frac{(1+\lambda_1\lambda_2)(1-\lambda_1\lambda_2)}{\left(1-\lambda_1^2\right)(1-\lambda_1\lambda_2)^2\left(1-\lambda_2^2\right)}\Omega_\varepsilon \\
&= \frac{1+\lambda_1\lambda_2}{\left(1-\lambda_1^2\right)(1-\lambda_1\lambda_2)\left(1-\lambda_2^2\right)}\Omega_\varepsilon \qquad (189)
\end{aligned}$$

which matches equation 177.

In the case of complex conjugate eigenvalues $\overline{\lambda_2} = \lambda_1$, the numerator is $1-|\lambda_1|^2|\lambda_2|^2 = \left(1+|\lambda_1|^2\right)\left(1-|\lambda_1|^2\right)$, and the denominator is $\left(1-|\lambda_1|^2\right)\left(1-\lambda_1^2\right)\left(1-\overline{\lambda_1}^2\right)\left(1-|\lambda_1|^2\right)$. Therefore:

$$\begin{aligned}
\Omega_y(\infty) &= \frac{\left(1+|\lambda_1|^2\right)\left(1-|\lambda_1|^2\right)}{\left(1-|\lambda_1|^2\right)\left(1-\lambda_1^2\right)\left(1-\overline{\lambda_1}^2\right)\left(1-|\lambda_1|^2\right)}\Omega_\varepsilon \\
&= \frac{1+|\lambda_1|^2}{\left(1-\lambda_1^2\right)\left(1-|\lambda_1|^2\right)\left(1-\overline{\lambda_1}^2\right)}\Omega_\varepsilon \\
&= \frac{1+\lambda_1\overline{\lambda_1}}{\left(1-\lambda_1^2\right)\left(1-\lambda_1\overline{\lambda_1}\right)\left(1-\overline{\lambda_1}^2\right)}\Omega_\varepsilon \\
&= \frac{1+\lambda_1\lambda_2}{\left(1-\lambda_1^2\right)(1-\lambda_1\lambda_2)\left(1-\lambda_2^2\right)}\Omega_\varepsilon \qquad (190)
\end{aligned}$$

which, with $\lambda_2 = \overline{\lambda_1}$, matches equation 177.

The FEV for the AR(2) is best expressed relative to $\Omega_y(\infty)$. That is, using the identity $\Omega_X^*(H) = \Omega_X^*(\infty) + [\Omega_X^*(H) - \Omega_X^*(\infty)]$ gives:

$$\begin{aligned}
JV\Omega_X^*(H)\,V^\dagger J' &= JV\left(\Omega_X^*(\infty) + [\Omega_X^*(H) - \Omega_X^*(\infty)]\right)V^\dagger J' \\
&= JV\Omega_X^*(\infty)\,V^\dagger J' + JV[\Omega_X^*(H) - \Omega_X^*(\infty)]\,V^\dagger J' \\
&= \Omega_y(\infty) + JV[\Omega_X^*(H) - \Omega_X^*(\infty)]\,V^\dagger J' \qquad (191)
\end{aligned}$$

where the individual elements of $\left[\Omega_X^*(H)-\Omega_X^*(\infty)\right]_{ij}$ are calculated as

$$\left[\Omega_X^*(H)-\Omega_X^*(\infty)\right]_{ij}=-\Omega_{E_X,ij}^*\frac{\left(\lambda_i\overline{\lambda_j}\right)^H}{1-\lambda_i\overline{\lambda_j}} \tag{192}$$

which gives the result:

$$\begin{aligned}
\left[\Omega_X^*(H)-\Omega_X^*(\infty)\right] &= -\Omega_{E_X}^*\circ\begin{bmatrix}\frac{|\lambda_1|^{2H}}{1-|\lambda_1|^2} & \frac{\left(\lambda_1\overline{\lambda_2}\right)^H}{1-\lambda_1\overline{\lambda_2}}\\ \frac{\left(\overline{\lambda_1}\lambda_2\right)^H}{1-\overline{\lambda_1}\lambda_2} & \frac{|\lambda_2|^{2H}}{1-|\lambda_2|^2}\end{bmatrix}\\
&= -\frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\circ\begin{bmatrix}\frac{|\lambda_1|^{2H}}{1-|\lambda_1|^2} & \frac{\left(\lambda_1\overline{\lambda_2}\right)^H}{1-\lambda_1\overline{\lambda_2}}\\ \frac{\left(\overline{\lambda_1}\lambda_2\right)^H}{1-\overline{\lambda_1}\lambda_2} & \frac{|\lambda_2|^{2H}}{1-|\lambda_2|^2}\end{bmatrix}\\
&= -\frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\begin{bmatrix}\frac{|\lambda_1|^{2H}}{1-|\lambda_1|^2} & -\frac{\left(\lambda_1\overline{\lambda_2}\right)^H}{1-\lambda_1\overline{\lambda_2}}\\ -\frac{\left(\overline{\lambda_1}\lambda_2\right)^H}{1-\overline{\lambda_1}\lambda_2} & \frac{|\lambda_2|^{2H}}{1-|\lambda_2|^2}\end{bmatrix}
\end{aligned} \tag{193}$$

Therefore:

$$\begin{aligned}
& JV\left[\Omega_X^*(H)-\Omega_X^*(\infty)\right]V^\dagger J'\\
&= -\frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\begin{bmatrix}\lambda_1 & \lambda_2\end{bmatrix}\begin{bmatrix}\frac{|\lambda_1|^{2H}}{1-|\lambda_1|^2} & -\frac{\left(\lambda_1\overline{\lambda_2}\right)^H}{1-\lambda_1\overline{\lambda_2}}\\ -\frac{\left(\overline{\lambda_1}\lambda_2\right)^H}{1-\overline{\lambda_1}\lambda_2} & \frac{|\lambda_2|^{2H}}{1-|\lambda_2|^2}\end{bmatrix}\begin{bmatrix}\overline{\lambda_1}\\ \overline{\lambda_2}\end{bmatrix}\\
&= -\frac{\Omega_\varepsilon}{|\lambda_1-\lambda_2|^2}\left(\frac{\left(\lambda_1\overline{\lambda_1}\right)^{H+1}}{1-\lambda_1\overline{\lambda_1}}-\frac{\left(\lambda_1\overline{\lambda_2}\right)^{H+1}}{1-\lambda_1\overline{\lambda_2}}-\frac{\left(\overline{\lambda_1}\lambda_2\right)^{H+1}}{1-\overline{\lambda_1}\lambda_2}+\frac{\left(\lambda_2\overline{\lambda_2}\right)^{H+1}}{1-\lambda_2\overline{\lambda_2}}\right)
\end{aligned} \tag{194}$$

E Accommodating repeated eigenvalues

This appendix discusses how repeated eigenvalues may be accommodated when applying the EAR framework. Section E.1 provides an overview of the eigensystem decomposition $\Phi=V\Lambda V^{-1}$ when repeated eigenvalues are included in the AR(P) specification, and section E.2 provides the standard algebraic expressions used to obtain V, Λ, and Λ^h when general case of repeated eigenvalues. Section E.3 provides the specific example of how a single pair of repeated eigenvalues is accommodated within the closed-form forecasts and decompositions for an AR(P).

E.1 Overview

The core of applying the EAR framework to closed-form AR(P) forecast/IRF components, historical component decomposition, and closed-form FEV and ergodic variances is the eigensystem decomposition of the AR(P) companion form, i.e. $\Phi=V\Lambda V^{-1}$, and its powers, i.e. $\Phi^h=\left(V\Lambda V^{-1}\right)^h=V\Lambda^hV^{-1}$ (or equivalently $\Phi=V_X\Lambda V_X^{-1}$ and its powers $\Phi^h=V_X\Lambda^hV_X^{-1}$). For example, the forecast/IRF expression $\mathbb{E}_t\left[y_{t+h}\right]=J\Phi^hY_t$ in section 5.2 leads to the forecast/IRF components $\mathbb{E}_t\left[y_{t+h}\right]=[1,\ldots,1]\,\Lambda^hX_t$ with

$X_t = \Lambda^{P-1} V^{-1} Y_t$. Analogously, the terms $J\Phi^h J' \Omega_\varepsilon J \left(\Phi^h\right)' J'$ from the FEV summation in section 5.4 with $\Phi^h = V\Lambda^h V^{-1}$, or equivalently $\Phi^h = V_X \Lambda^h V_X^{-1}$, leads to closed-form geometric sums for the elements $[\Omega_X(H)]_{ij}$ based on the eigenvalues λ_i and λ_j.

When an EAR is specified and estimated with repeated eigenvalues, V and Λ need to be altered from their purely diagonal form that applies in the case of distinct eigenvalues. In particular, Λ is specified with Jordan blocks associated with the repeated eigenvalues, and the associated eigenvectors in V are also adjusted using standard algebriac forms that will be discussed in the following section. The other eigenvalues in Λ remain in purely diagonal form, and their associated eigenvectors remain in the Vandermonde form already presented for distinct eigenvalues.

Applying the AR(P) eigensystem to forecasts/IRFs, historical components, and FEV and ergodic variances therefore requires taking the powers of the Jordan block/s within Λ along with powers of the purely diagonal part of the Λ. The diagonal case is very straightforward, given Λ^h simply produces scalar powers of the eigenvalues, i.e. $\Lambda^h = \text{diag}([\lambda_1, \ldots, \lambda_P])^h = \text{diag}\left([\lambda_1^h, \ldots, \lambda_P^h]\right)$, and this result also applies to the purely diagonal component of Λ. Conversely, powers of Jordan blocks need to account for their $(j, j+1)$ elements of 1, which results in upper triangular entries containing functions that are not simply scalar powers of the eigenvalues. Section E.2 provides the standard algebriac forms for powers of Jordan blocks.

Once V, Λ, and Λ^h have been altered to account for the repeated eigenvalue specification, they are used in the expressions for forecasts/IRFs, historical components, and FEV and ergodic variances. The end-result is that their components or elements associated with distinct eigenvalues will remain as for the purely diagonal cases presented in section 5, while the repeated eigenvalues will produce components with additional functional forms. These additional functions are illustrated in section E.3 for the forecast/IRF function in the case for a single pair of repeated eigenvalues, which applies to the empirical examples in section 6.

E.2 General repeated eigenvalues

With a single set of r repeated eigenvalues, the first r columns of V and the top-left $r \times r$ submatrix of Λ would be replaced with:[30]

$$V_r = \begin{bmatrix} \lambda_1^{P-1} & \binom{P-1}{1}\lambda_1^{P-2} & \cdots & \binom{P-1}{r-1}\lambda_1^{P-r} \\ \lambda_1^{P-2} & \binom{P-2}{1}\lambda_1^{P-3} & \cdots & \binom{P-2}{r-1}\lambda_1^{P-r-1} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^2 & 2\lambda_1 & \cdots & 0 \\ \lambda_1 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}; \Lambda_r = \begin{bmatrix} \lambda_1 & 1 & & \mathbf{0} \\ & \lambda_1 & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & \lambda_1 \end{bmatrix} \quad (195)$$

where the binomial coefficients for generic n and k, which are integers with $n \geq k$, are defined as:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \quad (196)$$

with "!" the factorial operator, e.g. $n! = n \times (n-1) \times \ldots \times 2 \times 1$.

[30] See Wilkinson (1965) pp. 14-15.

The eigenvalue matrix Λ is block diagonal, i.e.:

$$\Lambda = \text{diag}\left([\Lambda_r, \lambda_{r+1}, \ldots, \lambda_P]\right) \tag{197}$$

where $\lambda_{r+1}, \ldots, \lambda_P$ are the distinct eigenvalues. $\Lambda^h = \text{diag}\left(\left[\Lambda_r^h, \lambda_3^h, \ldots, \lambda_P^h\right]\right)$ will therefore also be block diagonal with Λ_r^h as follows:[31]

$$\Lambda_r^h = \begin{bmatrix} \lambda_1^h & \binom{h}{1}\lambda_1^{h-1} & \binom{h}{2}\lambda_1^{h-2} & \cdots & \binom{h}{r-1}\lambda_1^{h-r+1} \\ 0 & \lambda_1^h & \binom{h}{1}\lambda_1^{h-1} & \cdots & \binom{h}{r-1}\lambda_1^{h-r+1} \\ 0 & 0 & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \binom{h}{1}\lambda_1^{h-1} \\ 0 & 0 & 0 & \cdots & \lambda_1^h \end{bmatrix} \tag{198}$$

The component of $\mathbb{E}_t\left[y_{t+h}\right]$ associated with the eigenvalues $\lambda_1, \ldots, \lambda_r$, which I will denote as $\mathbb{E}_t\left[y_{t+h}|\lambda_1, \ldots, \lambda_r\right]$, is:

$$\mathbb{E}_t\left[y_{t+h}|\lambda_1, \ldots, \lambda_r\right] = JV_r\Lambda_r^h \begin{bmatrix} \left[V^{-1}Y_t\right]_1 \\ \vdots \\ \left[V^{-1}Y_t\right]_r \end{bmatrix} \tag{199}$$

and the component of $\mathbb{E}_t\left[y_{t+h}\right]$ associated with the distinct eigenvalues is:

$$\begin{aligned} \mathbb{E}_t\left[y_{t+h}|\lambda_{r+1}, \ldots, \lambda_P\right] &= \left[\lambda_{r+1}^h, \ldots, \lambda_P^h\right] X_t \\ &= \sum_{k=r+1}^{P} \lambda_k X_{k,t} \end{aligned} \tag{200}$$

The case of two or more sets of repeated eigenvalues combines the forms already given for the single set of repeated eigenvalues. For example, with a set of r repeated eigenvalues equal to λ_1 and a set of s repeated eigenvalues equal to λ_2, the first $r+s$ columns of V and the top-left $(r+s) \times (r+s)$ submatrix of Λ would respectively be:

$$V_{r+s} = [V_r, V_s] \; ; \; \Lambda_{r+s} = \text{diag}\left([\Lambda_r, \Lambda_s]\right) \tag{201}$$

where V_r is the $P \times r$ matrix in equation 195, V_s is the analogous $P \times s$ matrix using λ_2, Λ_r is the $r \times r$ matrix in equation 195, and Λ_s is the analogous $s \times s$ matrix using λ_2. The eigenvalue matrix is therefore $\Lambda = \text{diag}([\Lambda_r, \Lambda_s, \lambda_{r+s+1}, \ldots, \lambda_P])$, its powers are $\Lambda^h = \Lambda = \text{diag}\left(\left[\Lambda_r^h, \Lambda_s^h, \lambda_{r+s+1}^h, \ldots, \lambda_P^h\right]\right)$, Λ_r^h is given in equation 198, and Λ_s^h is the analogous result using λ_2. The component of $\mathbb{E}_t\left[y_{t+h}\right]$ associated with the eigenvalues $\lambda_1, \ldots, \lambda_{r+s}$ is:

$$\mathbb{E}_t\left[y_{t+h}|\lambda_1, \ldots, \lambda_{r+s}\right] = JV_{r+s}\Lambda_{r+s}^h \begin{bmatrix} \left[V^{-1}Y_t\right]_1 \\ \vdots \\ \left[V^{-1}Y_t\right]_{r+s} \end{bmatrix} \tag{202}$$

and the component of $\mathbb{E}_t\left[y_{t+h}\right]$ associated with the distinct eigenvalues is:

$$\begin{aligned} \mathbb{E}_t\left[y_{t+h}|\lambda_{r+s+1}, \ldots, \lambda_P\right] &= \left[\lambda_{r+s+1}^h, \ldots, \lambda_P^h\right] X_t \\ &= \sum_{k=r+s+1}^{P} \lambda_k X_{k,t} \end{aligned} \tag{203}$$

[31] See Hamilton (1994) p. 19.

E.3 AR(P) containing a single pair of repeated eigenvalues

Using equation 195, the case of a single pair of repeated eigenvalues would respectively replace the first two columns of V and the top-left 2×2 submatrix of Λ with:

$$V_2 = \begin{bmatrix} \lambda_1^{P-1} & (P-1)\,\lambda_1^{P-2} \\ \lambda_1^{P-2} & (P-2)\,\lambda_1^{P-3} \\ \vdots & \vdots \\ \lambda_1^2 & 2\lambda_1 \\ \lambda_1 & 1 \\ 1 & 0 \end{bmatrix} ; \Lambda_2 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \tag{204}$$

The eigenvalue matrix Λ will now be block diagonal, i.e. $\Lambda = \text{diag}([\Lambda_2, \lambda_3, \ldots, \lambda_P])$, where $\lambda_3, \ldots, \lambda_P$ are distinct eigenvalues. Λ^h will therefore also be block diagonal, i.e.:

$$\Lambda^h = \text{diag}\left(\left[\Lambda_2^h, \lambda_3^h, \ldots, \lambda_P^h\right]\right) \tag{205}$$

with Λ_2^h as follows:

$$\Lambda_2^h = \begin{bmatrix} \lambda_1^h & h\lambda_1^{h-1} \\ 0 & \lambda_1^h \end{bmatrix} \tag{206}$$

Following the method for the proof of Proposition 4, the forecast/IRF for $\mathbb{E}_t\left[y_{t+h}\right]$ is:

$$\begin{aligned} \mathbb{E}_t\left[y_{t+h}\right] &= J\left(V\Lambda V^{-1}\right)^h Y_t \\ &= JV\Lambda^h V^{-1} Y_t \\ &= JV\Lambda^h \Lambda^{1-P}\Lambda^{P-1} V^{-1} Y_t \\ &= JV\Lambda^{h+1-P} X_t \end{aligned} \tag{207}$$

where $X_t = \Lambda^{P-1} V^{-1} Y_t$.

The distinct eigenvalues $\lambda_3^h, \ldots, \lambda_P^h$ contribute to $\mathbb{E}_t\left[y_{t+h}\right]$ as in Proposition 4, i.e.:

$$\begin{aligned} \mathbb{E}_t\left[y_{t+h} | \lambda_3, \ldots, \lambda_P\right] &= \left[\lambda_3^h, \ldots, \lambda_P^h\right] X_t \\ &= \sum_{k=3}^{P} \lambda_k X_{k,t} \end{aligned} \tag{208}$$

The repeated eigenvalues $\lambda_1 = \lambda_2$ contribute to $\mathbb{E}_t\left[y_{t+h}\right]$ as:

$$\begin{aligned} \mathbb{E}_t\left[y_{t+h} | \lambda_1, \lambda_2\right] &= JV_2\Lambda_2^{h+1-P} \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1^{P-1} & (P-1)\,\lambda_1^{P-2} \end{bmatrix} \begin{bmatrix} \lambda_1^{h+1-P} & h\lambda_1^{h-P} \\ 0 & \lambda_1^{h+1-P} \end{bmatrix} \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1^h & (h+P-1)\,\lambda_1^{h-1} \end{bmatrix} \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} \\ &= \lambda_1^h X_{1,t} + (h+P-1)\,\lambda_1^{h-1} X_{2,t} \end{aligned} \tag{209}$$